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NUMERICAL COMPUTATION OF HYPERSONIC FLOW PAST A TWO-DIMENSIONAL BLUNT BODY

by

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August 15, 1964

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## ABSTRACT

Given a two-dimensional blunt body in a steady supersonic flow of compressible fluid, the problem is to study the flow between the body and the detached shock wave formed in front of it. In this paper, the shock shape is assumed to be analytic and we use it as an initial line in solving a Cauchy problem for the flow and the body. Hyperbolicity of the partial differential equations of motion is achieved in the subsonic region by means of complex extension on one of the independent variables. By means of such a substitution the first order system of nonlinear equations in two dimensions is transformed into a symmetric hyperbolic system in three independent variables. Such an increase in the number of independent variables can be avoided if the equations are written in normal form with respect to two characteristic coordinates. Difference schemes of second order accuracy are derived for both formulations of the complex extension and they are programmed for the IBM 7094. Five digit accuracy is obtained in the solution and numerical results are presented for a particular flow at Mach 6. The relative merits of the two formulations are discussed. Use of the three-dimensional symmetric hyperbolic system is simple because the equations are valid in the entire domain of solution.

However, stability considerations restrict the mesh ratios so that the body is barely reached. Less trouble is encountered in solving past the body when the equations in normal form are used. A serious drawback in this second approach is that the characteristics cusp at the sonic line. Thus, not only is one prevented from integrating across the sonic line, but also a portion of the flow is sometimes made inaccessible when characteristic coordinates are used.

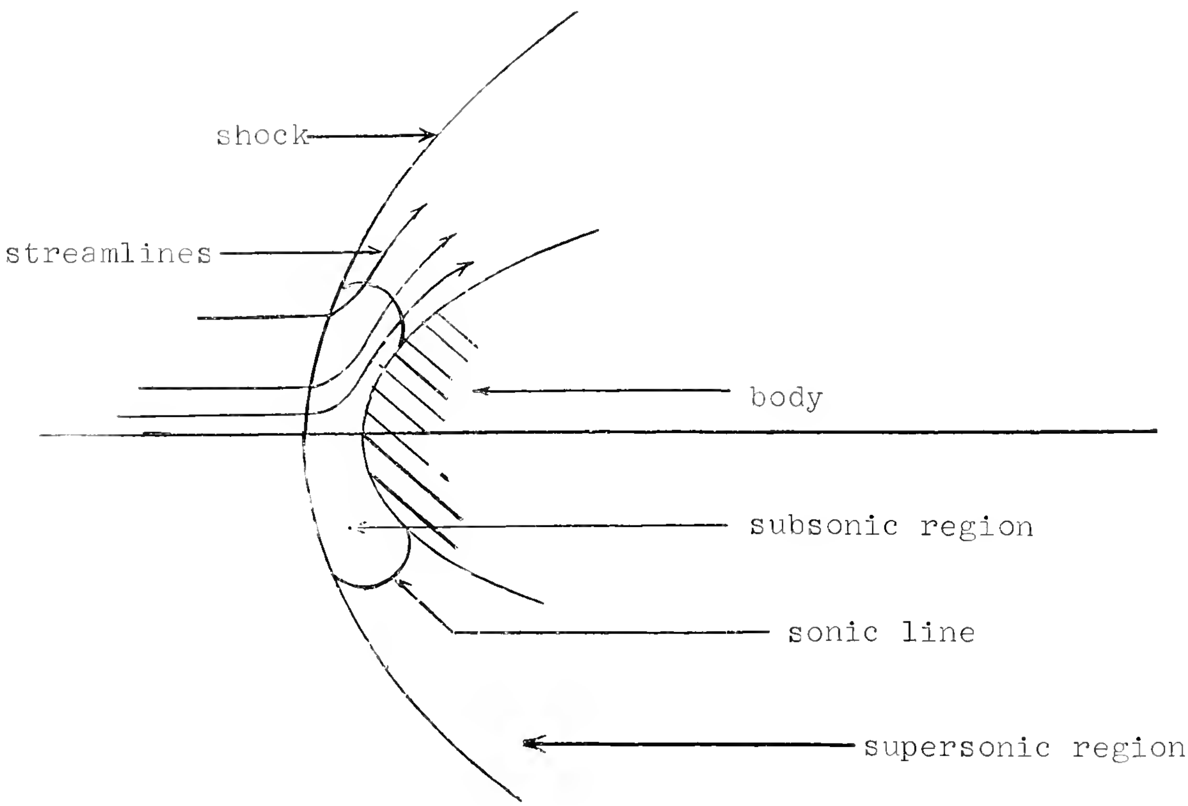
# NUMERICAL COMPUTATION OF HYPERSONIC FLOW PAST A TWO-DIMENSIONAL BLUNT BODY

by

Eva V. Swenson

## 1. Introduction

In this paper we shall consider a two-dimensional blunt body in a steady supersonic flow of compressible fluid. Our aim is to study the behavior of the flow between the body and the shock wave caused by the body. The direct approach consists in prescribing the body shape and solving the partial differential equations of motion of the flow. However, the



problem is complicated by the fact that the location and shape of the shock wave determined by the given body are unknown. An alternative approach, referred to as the inverse method, is to start from a known shock shape and to find the flow and the generating body by solving a Cauchy problem for the equations of motion. In the region where the flow is supersonic, the Cauchy problem is properly set because the equations of motion are of hyperbolic type. However, near the vertex of the shock wave, where it is known that the flow is subsonic, the partial differential equations of motion are elliptic and initial data on the shock wave are not appropriate to determine a solution.

Various methods have been applied to overcome the difficulty just described, including the use of double power series expansions and smoothing processes; see Lewis [10], and Hayes and Probstein [8], which contain a good exposition of the work to be done in these directions. In our case, we choose instead to follow the procedure described by Garabedian [6,7], which exploits the analyticity of the problem to continue the equations into the complex domain. The use of complex extension on one of the independent variables leads to hyperbolicity of the equations and the Cauchy problem becomes properly posed. One version of this method based on characteristic coordinates has been applied to the problem of three-dimensional flow past a blunt body of revolution [7]. For that case, however, difficulties were experienced along the sonic line.

For two-dimensional flow, we shall present in Sections 2 and 3 three possible formulations of the problem, the third of

which is a first order system with the velocity components $u$, $v$ and the stream function $\psi$ as dependent variables. We find that the latter formulation eliminates the difficulties previously encountered in solving for the flow in the transonic region. In Sections 4 and 5, complex extension is applied to the $u$, $v$, $\psi$-version in its two forms - both as a first order system and as a system in characteristic normal form.

Difference schemes are devised for the numerical solution of the two formulations of the problem, and their stability is discussed in Section 6. Both of these schemes have been programmed for the IBM 7094, and in Section 7 we evaluate the numerical results for a specific flow at Mach 6. The stability and convergence of the numerical scheme in practice is demonstrated in the tables. We have found that an accuracy of five significant figures can be achieved readily in the numerical calculations. Moreover, the results are in satisfactory agreement with the experimental data presently available on two dimensional flows past blunt bodies, see [1, 9, 11].

The author would like to express deepest gratitude to Professor Paul R. Garabedian who suggested the problem and who has been an invaluable source of inspiration, encouragement, and advice.

## 2. Equations of Motion

We shall start with the equations of motion for steady plane flow of a perfect compressible, inviscid fluid as derived in [3]. Let $x,y$ be Cartesian coordinates in the plane. We shall denote the horizontal and vertical components of the velocity

by $u(x,y)$ and $v(x,y)$, the pressure by $p(x,y)$, and the density by $\rho(x,y)$.

That mass is conserved is expressed by the continuity equation

$$(2.1) \qquad \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad .$$

Since pressure is the only force acting on the fluid, the conservation of momentum in the $x$ and $y$ directions, respectively, is expressed by

$$(2.2) \qquad u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + \frac{1}{\rho}\,\frac{\partial p}{\partial x} = 0 \quad ,$$

$$(2.3) \qquad u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + \frac{1}{\rho}\,\frac{\partial p}{\partial y} = 0 \quad .$$

From the law of conservation of energy we get the fourth equation of motion

$$(2.4) \qquad u\,\frac{\partial S}{\partial x} + v\,\frac{\partial S}{\partial y} = 0 \quad ,$$

which states that if the work done is due solely to the pressure forces, then the specific entropy $S$ at a moving particle remains constant. If we assume that our fluid satisfies the equation of state for a polytropic gas

$$(2.5) \qquad p = A(S)\rho^{\gamma} \quad , \qquad \gamma \text{ constant} \ ,$$

which asserts that $p\rho^{-\gamma}$ is a function of entropy alone, then we can rewrite (2.4) as

$$(2.6) \qquad u \frac{\partial}{\partial x} (p\rho^{-\gamma}) + v \frac{\partial}{\partial y} (p\rho^{-\gamma}) = 0 \ .$$

The constant $\gamma$ is the ratio of specific heats of our fluid.

Equations (2.1), (2.2), (2.3) and (2.6) together with relevant boundary conditions are sufficient to define the motion of a compressible fluid. We shall often use the conservation of energy equation in the integrated form

$$(2.7) \qquad \frac{1}{2} (u^2 + v^2) + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = H \ ,$$

where $H$ is a constant. Equation (2.7) is referred to as Bernoulli's equation.

A more convenient form of the equation of state (2.5) can be arrived at if we note that the continuity equation (2.1) implies the existence of a stream function $\psi(x,y)$ such that

$$(2.8) \qquad \psi_y = u\rho \ , \qquad - \psi_x = v\rho \ ,$$

that is, a function $\psi$ which is constant along each particle path, or streamline. Thus, we can rewrite (2.5) as

$$(2.9) \qquad p = A(\psi)\rho^{\gamma} \ ,$$

where $A$ can be considered as an arbitrary function of $\psi$ alone.

Our next task is to pick a suitable set of equations for our problem. The most obvious choice is the first order system for $u,v,p,\rho$ consisting of (2.1), (2.2), (2.3) and (2.6), which, written in matrix form, is

$$(2.10)\qquad \begin{pmatrix} u \\ v \\ p \\ \rho \end{pmatrix}_x = \begin{pmatrix} \frac{uv\rho}{D} & -\frac{\gamma p}{D} & -\frac{v}{D} & 0 \\ 0 & -\frac{v}{u} & -\frac{1}{u\rho} & 0 \\ -\frac{\gamma v p \rho}{D} & \frac{\gamma u p \rho}{D} & \frac{uv\rho}{D} & 0 \\ -\frac{v\rho^2}{D} & \frac{u\rho^2}{D} & \frac{v\rho}{uD} & -\frac{v}{u} \end{pmatrix} \begin{pmatrix} u \\ v \\ p \\ \rho \end{pmatrix}_y$$

where

$$D = \gamma p - u^2\rho = \rho(c^2 - u^2) \quad ,$$

and $c = \sqrt{dp/d\rho}$ is defined as the speed of sound. There is a considerable disadvantage in using this system. Note that it does not take into account the fact that the conservation of energy equation (2.6) can be integrated directly to give Bernoulli's equation (2.7). Not only does (2.6) add to the order of the system (2.10) unnecessarily, but it also introduces the streamlines as a family of characteristics. This is not desirable because near the body the streamlines bend in a direction parallel to the body and perpendicular to the direction in which we wish to integrate the system (2.10). We can never expect to get near the body with this formulation. Hence we intend to retain this

version only for checking purposes.

A second way of formulating the problem is motivated by the desire to solve for $\psi$ explicity. The stream function $\psi$ is of interest to us because the streamline $\psi = 0$ will serve to locate the body generating a given shock wave. From equations (2.2), (2.3), (2.7), and (2.9) one can derive (see [3,7]) the following second order quasilinear partial differential equation for $\psi$ :

$$(2.11) \quad (c^2-u^2)\psi_{xx}-2uv\,\psi_{xy}+(c^2-v^2)\psi_{yy}+\rho^2c^2\,\frac{A'(\psi)}{A(\psi)}\,\frac{2H+u^2+v^2}{2\gamma} = 0 .$$

Together with equations (2.8) and Bernoulli's equation (2.7) in the form

$$(2.12) \quad \frac{1}{2}\,\rho^{-2}\,(\psi_x^2+\psi_y^2)+\frac{\gamma}{\gamma-1}\,A(\psi)\,\rho^{\gamma-1} = H \quad ,$$

(2.11) gives an alternate equation for our problem. From equation (2.11) and (2.8), a first order system of partial differential equations in the dependent variables $\psi$ , $\psi_x$ and $\psi_y$ can be obtained.

Superficially, it might seem as though we have reduced our problem to that of solving three differential equations instead of four. However, the work of integrating a fourth differential equation has been replaced by that of solving for $\rho$ as a root of Bernoulli's equation (2.12). By applying Newton's method, we can obtain $\rho$ to second order accuracy. However,

at the sonic line, we have $u^2 + v^2 = c^2$ and therefore

$$H_\rho = -\rho^{-3} (\psi_x^2 + \psi_y^2) + \gamma A \rho^{\gamma-2}$$
$$= - \frac{u^2 + v^2}{\rho} + \frac{c^2}{\rho} = 0 \quad ,$$

so that Newton's method is no longer applicable and we have to resort to more sophisticated schemes.

In order to circumvent this difficulty, we introduce the set of dependent variables $\psi$ , $u$ , $v$ instead of $\psi$ , $\psi_x$ , $\psi_y$ . This brings Bernoulli's equation (2.12) into a form that is solvable for $\rho$ explicitly as

$$\rho = \left[ (H - \frac{u^2 + v^2}{2}) \frac{\gamma - 1}{\gamma A(\psi)} \right]^{1-\gamma} \quad .$$

We shall proceed to derive a set of equations for $u$ , $v$ , and $\psi$ based on the latter relation.

If we differentiate (2.7) with respect to $x$ and $y$ , we get the equations

$$(2.13) \qquad uu_x + vv_x + \frac{c^2}{\gamma-1} \frac{A'}{A} \psi_x + \frac{c^2}{\rho} \rho_x = 0 \quad ,$$

$$(2.14) \qquad uu_y + vv_y + \frac{c^2}{\gamma-1} \frac{A'}{A} \psi_y + \frac{c^2}{\rho} \rho_y = 0 \quad .$$

Multiplying (2.13) by $u$ , (2.14) by $v$ , and adding, we obtain

$$(2.15) \qquad u^2 u_x + uvv_x + uvu_y + v^2 v_y + \frac{c^2}{\rho} (u\rho_x + v\rho_y) = 0 \quad ,$$

where (2.8) determines $\psi_x$ and $\psi_y$. Using (2.1) we then re-write (2.15) as

$$(2.16) \qquad (c^2 - u^2)\, u_x - uv(u_y + v_x) + (c^2 - v^2) v_y = 0 \quad .$$

This is the first equation of our system.

The second equation may be obtained by subtracting (2.2) from (2.13) to get

$$(2.17) \qquad vv_x - vu_y + \frac{c^2}{\gamma-1} \; \frac{A'}{A} \psi_x + \frac{c^2}{\rho} \rho_x - \frac{1}{\rho} p_x = 0 \quad .$$

We can simplify (2.17) if we note that

$$p_x = A' \psi_x \rho^\gamma + \gamma A \rho_x \rho^{\gamma-1} \quad , \qquad c^2 = \gamma A \rho^{\gamma-1} \quad .$$

Then (2.17) becomes

$$v(v_x - u_y) - \frac{c^2}{\gamma-1} \; \frac{A'}{A} v\rho + \frac{c^2}{\gamma} \; \frac{A'}{A} \; v\rho = 0 \quad ,$$

or

$$(2.18) \qquad v_x - u_y = \frac{c^2}{\gamma(\gamma-1)} \; \frac{A'}{A} \rho \; .$$

We would like to remark that one can derive this same equation by working with (2.14), (2.3) and

$$p_y = A' \psi_y \rho^\gamma + \gamma A \rho_y \rho^{\gamma-1}$$

instead. To complete the set, we may use either of the equations

$$(2.8) \qquad \psi_x = -v\rho \quad , \qquad \psi_y = u\rho \quad ,$$

depending only on our choice of the distinguished independent variable. Thus altogether, our system in the unknowns $u, v, \psi$ is

$$(2.19) \qquad \begin{pmatrix} u \\ v \\ \psi \end{pmatrix}_x = \begin{pmatrix} \frac{2uv}{c^2-u^2} & -\frac{c^2-v^2}{c^2-u^2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \psi \end{pmatrix}_y + \begin{pmatrix} Q\frac{uv}{c^2-u^2} \\ Q \\ -v\rho \end{pmatrix} ,$$

where

$$Q = \frac{c^2\rho}{\gamma(\gamma-1)} \frac{A'(\psi)}{A(\psi)} .$$

Taking Bernoulli's equation (2.7) into account, we have a suitable formulation of our flow problem.

We shall be interested ultimately in a Cauchy problem for the first order system (2.19) with data prescribed behind a shock wave whose equation has the form $x = f(y)$. Therefore let us consider the change of coordinates

$$\xi = y \quad , \qquad \zeta = x - f(y)$$

which has the effect of mapping the initial curve onto the line $\zeta = 0$ in the $\zeta, \xi$-plane. The system of equations (2.19) which in matrix notation has the form $U_x = RU_y + S$ is then transformed

into

$$U_\zeta = [I + f'(\xi)R]^{-1} R\, U_\xi + [I + f'(\xi)R]^{-1} S \quad ,$$

or more specifically,

$$\begin{pmatrix} u \\ v \\ \psi \end{pmatrix}_\zeta = \frac{1}{D} \begin{pmatrix} 2uv + f'(\xi)\,(c^2-v^2) & -(c^2-v^2) & 0 \\ c^2-u^2 & f'(\xi)(c^2-v^2) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \psi \end{pmatrix}_\xi$$

(2.20)

$$+ \frac{1}{D} \begin{pmatrix} Q[uv + f'(\xi)\,(c^2-v^2)] \\ Q[c^2-u^2 + uv\, f'(\xi)] \\ -\, v\rho D \end{pmatrix} ,$$

where

$$D = (c^2-u^2) + 2uv\, f'(\xi) + f'^2(\xi)\,(c^2-v^2) \quad ,$$

$$Q = \frac{c^2}{\gamma(\gamma-1)} \; \frac{A'(\psi)}{A(\psi)}\, \rho \quad .$$

This is one of the principal formulations of the equations of motion that we shall use for the numerical solution of the detached shock problem.

A more subtle approach which we shall have occasion to exploit is the reduction of equations (2.16), (2.18) and (2.8)

to normal form (see [4,6]), that is, a form where each equation involves a differentiation in only one characteristic direction. We shall confine our attention to equations (2.16) and (2.18), since (2.8) can be put in characteristic normal form easily. In matrix notation, (2.16) and (2.18) can be written as

$$(2.21)\qquad \begin{pmatrix} c^2-u^2 & -uv \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} -uv & c^2-v^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_y - \begin{pmatrix} 0 \\ Q \end{pmatrix} = 0 ,$$

where

$$Q = \frac{c^2}{\gamma(\gamma-1)} \frac{A'(\psi)}{A(\psi)} \rho .$$

Given initial values of the vector $\mathrm{col}(u,v) = w$ on an initial curve $\phi(x,y) = 0$ with $(\phi_x)^2 + (\phi_y)^2 \neq 0$, we are to determine the vector $w$ whose first derivatives satisfy a system of the form $Ew_x + Fw_y + J = 0$. It has been proven in [4] that a necessary and sufficient condition for the unique determination of all the first derivatives along $\phi(x,y) = 0$ is the non-vanishing of the characteristic determinant of the system

$$(2.22)\qquad |E\tau - F| = \begin{vmatrix} (c^2-u^2)\tau + uv & -uv\tau - (c^2-v^2) \\ 1 & \tau \end{vmatrix} \neq 0 ,$$

where $\tau = -\phi_x/\phi_y$ . If $\tau(x,y)$ is a real solution of $|E\tau - F| = 0$ , then the curves along which $\tau = dy/dx = \mathrm{const.}$

are called characteristic curves. In our case, the characteristic slopes are solutions of

$$(2.23)\qquad (c^2-u^2)\left(\frac{dy}{dx}\right)^2 + 2uv\,\frac{dy}{dx} + (c^2-v^2) = 0 \quad ,$$

that is,

$$(2.24)\qquad \frac{dy}{dx} = \tau_{\pm} = \frac{-uv \pm c\sqrt{u^2+v^2-c^2}}{c^2-u^2} \quad .$$

If we solve the system of linear homogeneous equations $\lambda(E\tau - F) = 0$ for the vector $\lambda = (\lambda_1,\lambda_2)$ , our original system $Ew_x + Fw_y + J = 0$ can be rewritten as the linear combination

$$\lambda E w_x + \lambda E\tau w_y + \lambda J = 0 \quad ,$$

or

$$(2.25)\qquad \lambda E(w_x + \tau w_y) + \lambda J = 0 \quad ,$$

where now the unknown $w$ is differentiated along one characteristic direction.

Carrying out the above procedure in more detail, we find that the components $\lambda_1$, $\lambda_2$ of the vector $\lambda$ satisfy the ratios

$$(2.26)\qquad \frac{\lambda_2}{\lambda_1} = -\,[(c^2-u^2)\tau + uv] = \frac{uv\tau + (c^2-v^2)}{\tau} \quad .$$

Setting $\lambda_1 = 1$ and $\lambda_2 = -[(c^2-u^2)\tau + uv]$ , we have that

$\lambda_1$ times (2.16) plus $\lambda_2$ times (2.18) gives

$$
(2.27)\qquad
\begin{aligned}
&(c^2-u^2)u_x + [(c^2-u^2)\tau]u_y - [(c^2-u^2)\tau + 2uv]v_x + (c^2-v^2)v_y \\
&\qquad = - Q[(c^2-u^2)\tau + uv] \quad .
\end{aligned}
$$

Using (2.26), equation (2.27) becomes

$$
(2.28)\qquad \tau(c^2-u^2)(u_x+\tau u_y) + (c^2-v^2)(v_x+\tau v_y) = - Q\tau[(c^2-u^2)\tau + uv] \ ,
$$

where $\tau$ is defined in (2.24).

Let $\alpha(x,y) = 0$ and $\beta(x,y) = 0$ be the characteristic curves with slopes $\tau_+$ and $\tau_-$ , respectively. Then

$$
\frac{\partial}{\partial\alpha} = \frac{\partial}{\partial x} x_\alpha + \frac{\partial}{\partial y} y_\alpha = x_\alpha\left(\frac{\partial}{\partial x} + \frac{y_\alpha}{x_\alpha} \frac{\partial}{\partial y}\right) = x_\alpha\left(\frac{\partial}{\partial x} + \tau_+ \frac{\partial}{\partial y}\right) .
$$

Similarly

$$
\frac{\partial}{\partial\beta} = x_\beta \left(\frac{\partial}{\partial x} + \tau_- \frac{\partial}{\partial y}\right) .
$$

Hence (2.28) can be rewritten as

$$
(2.29)\qquad
\begin{aligned}
\tau_+(c^2-u^2)u_\alpha + (c^2-v^2)v_\alpha &= - Q\, c \sqrt{u^2+v^2-c^2}\ y_\alpha \ , \\
\tau_-(c^2-u^2)u_\beta + (c^2-v^2)v_\beta &= Q\, c \sqrt{u^2+v^2-c^2}\ y_\beta \ .
\end{aligned}
$$

Equations (2.29) together with

$$y_\alpha - \tau_+ x_\alpha = 0 ,$$

(2.30)

$$y_\beta - \tau_- x_\beta = 0 ,$$

and (2.8) written as

(2.31) $$\psi_\beta + \rho v \, x_\beta - \rho u \, y_\beta = 0 ,$$

comprise a system in normal form equivalent to (2.19).

## 3. Shock Conditions and Initial Conditions

The inverse detached shock problem consists of starting from a given shock and solving the equations of motion for the flow and the generating body. In other words, we have to solve a Cauchy problem for the system (2.19), or equivalently (2.29 - 31), with initial data on the shock wave. In this section we shall indicate how initial data along a shock wave are calculated from the shock conditions.

Across the shock, pressure and density are discontinuous, but the conservation laws enable us to relate the values of these quantities on either side of the shock. Adopting the notation of [3], which contains a derivation of the shock conditions, we let $N$ and $L$ denote the normal and tangential components of the velocity vector with respect to the shock. We shall use the subcript "o" to indicate quantities in front of the shock, and "1"

to denote quantities behind the shock.

That mass is conserved across the shock is expressed by

$$(3.1) \qquad \rho_o N_o = \rho_1 N_1 = m ,$$

where $m$ is the mass flux across the shock surface.
Conservation of momentum gives the equations

$$(3.2) \qquad \rho_o N_o^2 + p_o = \rho_1 N^2 + p_1 = P , \quad L_o = L_1 ,$$

where $P$ is the total flux of momentum normal to the shock.
Finally we have

$$(3.3) \qquad \frac{1}{2} (N_o^2 + L_o^2) + \frac{\gamma}{\gamma-1} \frac{p_o}{\rho_o} = \frac{1}{2} (N_1^2 + L_1^2) + \frac{\gamma}{\gamma-1} \frac{p_1}{\rho_1} ,$$

which states that energy is conserved across the shock.

Given the quantities $L_o$ , $N_o$ , $p_o$ , $\rho_o$ in front of the shock, we shall compute the quantities $N_1$ , $p_1$ , $\rho_1$ behind the shock using the shock conditions (3.1 - 3). Since these equations are nonlinear in $N_1$ , $p_1$ , and $\rho_1$ , we have to use the auxiliary equations

$$(3.4) \qquad N_o N_1 = \frac{p_1 - p_o}{\rho_1 - \rho_o} ,$$

and

$$(3.5) \qquad p_1 = \rho_o N_o^2 (1-\mu^2) - \mu^2 p_o .$$

The first is referred to as Prandtl's relation; the second as Hugoniot's relation. Both may be derived by performing a few algebraic manipulations on the shock conditions (3.1 - 3).

Hence, given $\rho_o$, $p_o$, $N_o$, and $L_o$ in front of the shock, we can solve for $L_1$, $p_1$, $N_1$, and $\rho_1$ in the sequence

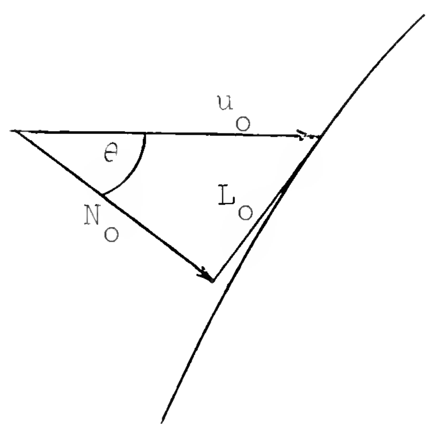
$$L_1 = L_o,$$

(3.6)
$$p_1 = \rho_o N_o^2(1-\mu^2) - \mu^2 p_o \quad ,$$

$$N_1 = [\mu^2 N_o^2 + (1-\mu^2)\gamma]/N_o \quad ,$$

$$\rho_1 = \rho_o N_o/N_1 \quad .$$

In our particular problem we shall take the direction of incident flow to be parallel to the axis of the body, which is itself parallel to the x-axis. Thus $v_o = 0$ and $u_o = U_o$, a constant in front of the shock. Let $\theta$ be the angle which the normal to the shock wave makes with the horizontal. Then with respect to the shock wave $x = f(y)$, we have



$$L_o = u_o \sin\theta = \frac{U_o f'(y)}{\sqrt{1+f'^2(y)}} \,, \tag{3.7}$$

$$N_o = u_o \cos\theta = \frac{U_o}{\sqrt{1+f'^2(y)}} \,.$$

Without loss of generality, we shall choose a scale so that $\rho_o = p_o = 1$. The speed of a flow is usually specified by giving its Mach number $M$ defined as

$$M = \frac{\sqrt{u^2+v^2}}{c} \,, \tag{3.8}$$

where $\sqrt{u^2+v^2}$ is the speed of the flow and $c$ is the local speed of sound. In our case $c_o^2 = \gamma p_o/\rho_o = \gamma$, hence $u_o = U_o = c_o M = \sqrt{\gamma}\, M$.

Using the equations (3.6) we then have expressions for $L_1$, $N_1$, $p_1$, $\rho_1$ compatible with our conditions on the incident flow. They are as follows:

$$L_1 = \frac{\sqrt{\gamma}\, M\, f'(y)}{\sqrt{1+f'^2(y)}} \,,$$

$$p_1 = \frac{(1-\mu^2)\,\gamma\, M^2}{1+f'^2(y)} - \mu^2 \,,$$

$$N_1 = \left[\frac{2\gamma}{\gamma+1} + \frac{\mu^2\,\gamma\, M^2}{1+f'^2(y)}\right] \frac{\sqrt{1+f'^2(y)}}{\sqrt{\gamma}\; M} \,, \tag{3.9}$$

$$\rho_1 = \frac{\gamma M^2}{1+f'^2(y)} \left[\frac{2\gamma}{\gamma+1} + \frac{\mu^2\,\gamma\, M^2}{1+f'^2(y)}\right]^{-1} \,.$$

Finally, we obtain $u_1$ , $v_1$ from the relations

$$(3.10) \qquad \begin{aligned} u_1 &= N_1 \cos\theta + L_1 \sin\theta , \\ v_1 &= L_1 \cos\theta - N_1 \sin\theta . \end{aligned}$$

At this point, we would like to note that the expressions for $u_1$ , $v_1$ , $p_1$ and $\rho_1$ consist of elementary and analytic formulas depending only on $f'(y)$. They are elementary in the sense that only the operations of addition, multiplication, division and square root are involved. If the expression $f(y)$ for the shock wave is similarly elementary and analytic, then the equations (3.9) may be continued analytically into the complex $\boldsymbol{x}$-plane by inspection, i.e. simply by substituting for $y$ $(=\xi)$ in (3.9) the complex variable $\xi + i\eta$ . That the initial data may be extended to complex values is a crucial property that we shall resort to later on when we solve the Cauchy problem in the complex domain.

## 4. Complex Extension Applied to Equations In Characteristic Normal Form

Thus far, we have a formulation of the inverse problem written in two ways, namely as a first order system (2.20) and as a system in characteristic normal form (2.29-31). In Section 3, we demonstrated how initial conditions behind the shock are obtained from the shock conditions. The partial

differential equations and initial conditions comprise the full statement of the Cauchy problem for the flow.

We shall concern ourselves at present in making sure that such a formulation is a proper one in the sense that the solution depends continuously on the initial data. In the supersonic region, the equations are of hyperbolic type and the prescribed initial conditions are sufficient for the existence of a unique solution. However, in the subsonic region, the equations are of elliptic type, which means that initial conditions do not specify a properly posed problem. Following the procedure used by Garabedian [2,6,7], we shall take advantage of the analyticity of the equations and apply the method of complex extension to them. The resulting equations in the complex domain will be of hyperbolic type, so that the Cauchy problem becomes well-posed there.

In this section, we shall concentrate our attention on the equations (2.29-31) in characteristic normal form. For easy reference, we restate them here as

$$y_\alpha - \tau_+ x_\alpha = 0 , \qquad y_\beta - \tau_- x_\beta = 0 ,$$

$$\tau_+(c^2-u^2)u_\alpha + (c^2-v^2)v_\alpha + Qc\sqrt{u^2+v^2-c^2} \quad y_\alpha = 0,$$

(4.1)

$$\tau_-(c^2-u^2)u_\beta + (c^2-v^2)v_\beta - Qc\sqrt{u^2+v^2-c^2} \quad y_\beta = 0,$$

$$\psi_\beta + \rho v x_\beta - \rho u y_\beta = 0 \quad ,$$

where

$$\tau_{\pm} = \frac{-uv \pm c\sqrt{u^2+v^2-c^2}}{c^2-u^2} \quad .$$

We shall limit our considerations to the subsonic region where the Cauchy problem is not well-posed.

In the subsonic region, $u^2+v^2 < c^2$ and the solutions of the characteristic equation (2.23) are in general complex. If $\alpha(x,y) =$ const. denotes one family of characteristics, where

$$\alpha(x,y) = t(x,y) + is(x,y) ,$$

then we must have

$$\beta(x,y) = t(x,y) - is(x,y) = \text{const.}$$

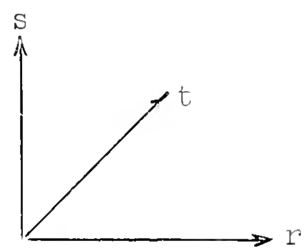
as the second family of characteristics because the coefficients of (2.23) are real when $x$ and $y$ are real. Now $\alpha =$ const. and $\beta =$ const. may serve as a suitable set of characteristic coordinates for the canonical system (4.1) in the subsonic region if $x,y,u,v,\psi$ are allowed to assume complex values. The system (4.1) will, however, be hyperbolic only when restricted to that coordinate system with respect to which the characteristic curves $\alpha =$ const. and $\beta =$ const. are real.

The extension of the flow quantities to complex functions of the complex variables $x$ and $y$ may be done by the basic rules of analytic continuation because of the analyticity

of these functions. Let us for the moment keep $x$ and $y$ real. In the real $x,y$-plane, $t$ and $s$ are real-valued functions. Since, in addition, they are conjugate harmonic functions, we can perform the transformation $(x,y) \longrightarrow (t,s)$ . Then the curves $\alpha$ and $\beta$ in the $x,y$-plane become the straight lines

$$\alpha(t,s) = t + is \quad , \quad \beta(t,s) = t - is \ ,$$

in the $t,s$-plane. For convenience, let us assume that the shock wave is mapped onto the $s$-axis. We then make a judicious use of complex extension and extend the variable $s$ to the complex plane so that




$$q = r + is \quad .$$

Then the corresponding extensions of $\alpha$ and $\beta$ are

$$\alpha(t,q) = (t+r) + is \quad ,$$

$$\beta(t,q) = (t-r) - is \quad .$$

If we restrict our attention to the $t,r$-plane, we observe that $\alpha$ and $\beta$ are real functions, i.e. in this plane the system (4.1) has real characteristics and thus, is hyperbolic.

The Cauchy data for this system is now the complex extension into the $r,s$-plane of the initial data that was defined for $t$ and $s$ real. No difficulties arise from the analytic continuation because, as we noted in Section 3, the initial data for $u,v,p,\rho$ are computed from analytical formulas (3.9 - 10) which involve only the basic operations of addition, subtraction, multiplication, division and square root, and depend on a function $f'(y)$ which is also analytic and elementary.

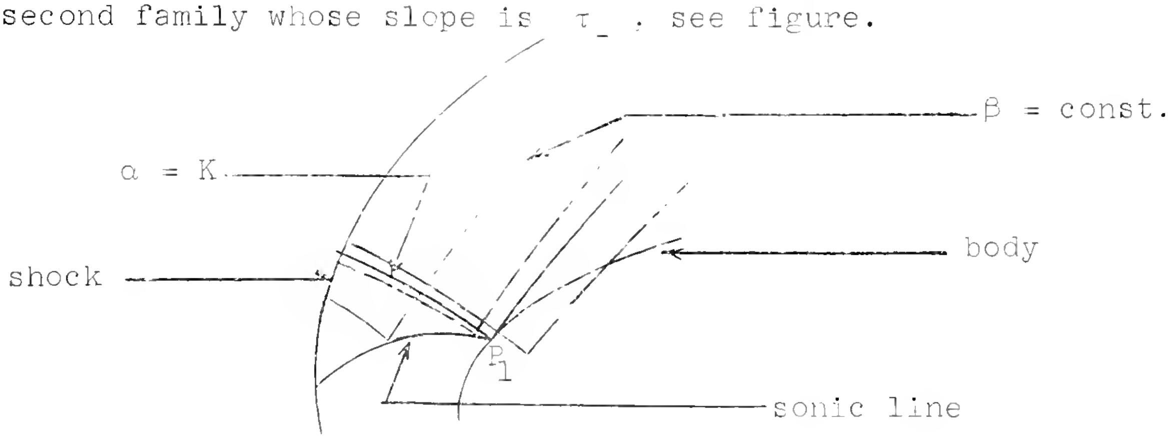
The solutions we obtain shall be valid in some section of the $t,r$-plane since $s$ is kept fixed as a parameter in our calculations. If in this surface of solution we set $r = 0$, then we obtain in the $t,s$-plane a line of real solutions which is of physical interest. To get the solution in a full region of the $t,s$-plane, we must then solve the hyperbolic problem repeatedly for a full range of values of the parameter $s$.

Since $s$ is a parameter, the $t,s$-plane may still be transformed conformally provided that the segment of the $s$-axis which contains the Cauchy data is preserved. Thus any analytic change of scale in $s$ may be introduced. This means that the initial values of any one of the unknowns, $x,y,u,v,\psi$, say $y$ can be given by a quite arbitrary analytic expression in $s$. Given a fixed $s$ then, this implies that $y$, as a complex-valued function of $r$, can be any complex-valued analytic expression that assumes its conjugate value at $-r$. In particular $y(r)$ may be any continuously differentiable function of $r$, to which we can approximate by analytic expressions in $r$. This is possible since we have a well-posed Cauchy problem and

any convergent limiting process on the data still leaves us with valid results. All the other variables $x,u,v,\psi$ must however be initially related to $y$ by the analytic conditions of our Cauchy problem in the original $x,y$-plane. In essence, our choice of the function $y = y(r)$ is equivalent to choosing a path of integration in the real $x,y$-plane if we were to solve the Cauchy problem by successive approximations. Since $y(r)$ need not be analytic, this path need not be analytic either. As we shall see, it can have corners, cf. Figs. 2,3.

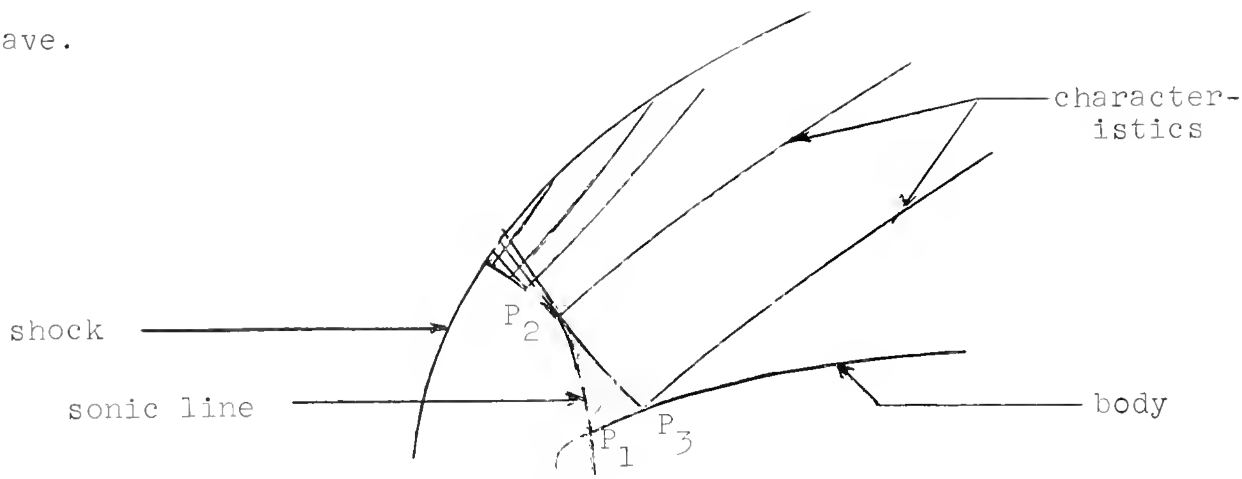
So far, we have indicated a way of solving the Cauchy problem separately in the subsonic and supersonic regions of the flow. This method is not completely satisfactory in the subsonic region because each calculation gives us only a line of solutions. Moreover, in the supersonic region, this method has a serious defect. The system (4.1) becomes singular along the sonic line since along it $u^2+v^2 = c^2$, i.e. the term $\sqrt{u^2+v^2-c^2}$ has its branch point on the sonic line. This implies that characteristics in the supersonic region terminate on the sonic line and form cusps.

Let $\alpha(x,y) = \text{const.}$ denote the family of characteristics whose slope is $\tau_+$ and $\beta(x,y) = \text{const.}$ denote the second family whose slope is $\tau_-$, see figure.



Let $P_1$ be the point where the sonic line crosses the body. Consider the constant $K$ such that the characteristic curve $\alpha(x,y) = K$ crosses the body, any curve $\alpha(x,y) = K + \varepsilon$ for $\varepsilon > 0$ also crosses the body, but the curve $\alpha(x,y) = K - \varepsilon$ terminates on the sonic line. If the characteristic curve $\alpha(x,y) = K$ hits the sonic line at $P_1$, then we know that we can solve for the flow in the entire supersonic domain.

However, if the curve $\alpha(x,y) = K$ terminates at a point $P_3$ on the body a distance away from $P_1$ (see figure below and also Figure 3 in the Appendix), then we know that a section of the supersonic region bounded by the sonic line $P_1P_2$, the body $P_1P_3$ and the curve $P_2P_3$ which is the limit of the characteristic $\alpha(x,y) = K + \varepsilon$, as $\varepsilon \to 0$, lies outside the domain of influence of our initial data on the shock wave.




It is hoped that using the second formulation of the Cauchy problem (2.20), we shall be able to solve for the flow in this hidden transonic region. We turn our attention to it in the next section.

## 5. Complex Extension Applied to a General First Order System in 2 Independent Variables

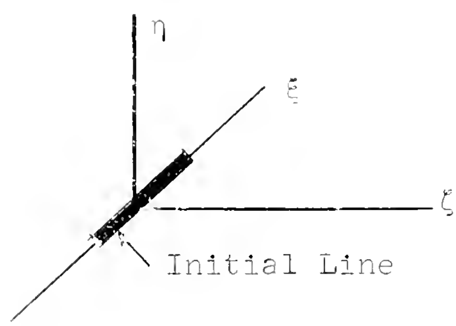
We shall concern ourselves with the system (2.20) in this section. In matrix notation, (2.20) has the form

$$U_\zeta = RU_\xi + S \ . \tag{5.1}$$

Together with some Cauchy data along the initial curve $\zeta = 0$, we know that we can solve this system by successive approximations wherever (5.1) is of hyperbolic type. However, as we saw in Section 4, this is not the case in the subsonic region.

By the use of complex extension on the independent variable $\xi$, we shall demonstrate how we can obtain a system equivalent to (5.1) which is hyperbolic in both the subsonic and supersonic regions. We proceed as follows: Consider $\xi$ as the real part of the complex variable $\chi$ so that $\chi = \xi + i\eta$. Using the reflection principle, we can likewise extend all the dependent quantities $u, v, p, \rho$ as complex functions of $\chi$. The system (5.1) then becomes

$$U_\zeta = RU_\chi + S \ . \tag{5.2}$$




Note that in general $U_\chi = U_\xi = -i\, U_\eta$, since by the Cauchy-Riemann equations

$$U_\xi = (\mathrm{Re}\ U)_\xi + i(\mathrm{Im}\ U)_\xi = (\mathrm{Im}\ U)_\eta - i(\mathrm{Re}\ U)_\eta = -\,iU_\eta .$$

Hence we may write $U_x = \frac{1}{2}(U_\xi - iU_\eta)$ . Similarly, if we write $\bar{x} = \xi - i\eta$ , then $U_{\bar{x}} = \frac{1}{2}(U_\xi + iU_\eta) = 0$ . We can thus add the term $R^* U_{\bar{x}}$ to the right hand side of (5.2) obtaining the system

$$U_\zeta = R\,U_x + R^*\,U_{\bar{x}} + B \quad , \tag{5.3}$$

where the asterisk * denotes the conjugate transpose of the matrix. Reverting to the variables $\xi$ , $\eta$ , and $\zeta$ , equation (5.3) can be written as

$$U_\zeta = \frac{R + R^*}{2}\,U_\xi + \frac{R - R^*}{2i}\,U_\eta + S \ . \tag{5.4}$$

This new system is symmetric hyperbolic because the coefficient matrices $(R + R^*)/2$ and $(R - R^*)/2i$ are Hermitian and hence have real eigenvalues. Thus in the new $\xi$, $\eta$, $\zeta$ - domain, the Cauchy problem is well posed.

Our derivation of the system (5.4) did not depend in any way on whether we were in the subsonic or supersonic region. We should then be able to solve the equations in the transonic region without any difficulty, especially in that section which would have been impossible to penetrate by the method of characteristics. The one disadvantage to this formulation is that we have to carry out our computations in a three-

dimensional domain in order to get a surface of solution in the real $\zeta$, $\xi$ - plane.

From the above discussion, it would seem as though the instability of the elliptic Cauchy problem, in the sense of Hadamard, has been eliminated by the application of a simple complex extension to one of the variables. This is not the case, however. We replaced the task of solving an unstable elliptic Cauchy problem in the real $\zeta$ , $\xi$ - plane with that of having to solve a well-posed hyperbolic Cauchy problem in the complex $\zeta$ , $x$ - plane plus having to continue the initial conditions into the complex domain. Solving the hyperbolic problem numerically is a stable process. However, instability enters in the analytic continuation of the Cauchy data into the complex domain because this process is equivalent to solving Laplace's equation with Cauchy data. Hence in general, we may not be able to carry out the analytic continuation.

The success of the method in our particular case depends on the fact that we have chosen simple analytic curves like hyperbolas and parabolas for the shock shape, enabling us to continue the Cauchy data into the complex domain in closed form. Thus we readily obtain the analyticity of the data in the large and we can investigate the singularities arising from their continuation explicitly.

## 6. Numerical Solution of the Cauchy Problem

In this section we shall present two numerical schemes, one for the characteristic equations (4.1) and another for the first order system (5.4).

The system (4.1) is of the form

$$y_\beta = \tau_- (x,y,u,v,\psi)\, x_\beta \quad ,$$

$$y_\alpha = \tau_+ (x,y,u,v,\psi)\, x_\alpha \quad ,$$

$$(6.1) \qquad A_1 (x,y,u,v,\psi)\, u_\alpha + B_1 (x,y,u,v,\psi)\, v_\alpha = C_1 (x,y,u,v,\psi),$$

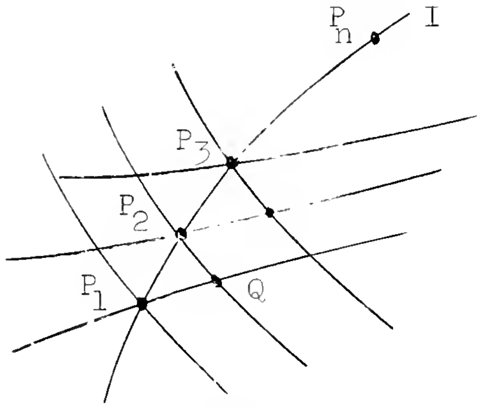
$$A_2 (x,y,u,v,\psi)\, u_\beta + B_2 (x,y,v,v,\psi)\, v_\beta = C_2 (x,y,u,v,\psi),$$

$$\psi_\beta = C_3 (x,y,u,v,\psi) \quad ,$$

and we shall be able to apply the method of Massau [5,13], which is designed specifically for a system of quasilinear hyperbolic equations in two independent variables. The system has to be in characteristic normal form because Massau's scheme takes advantage of the existence of two distinct families of characteristics.

Let $I$ be the initial non-characteristic curve and let $P_1$ , $P_2$ ,...,$P_n$ be a sequence of points along $I$ . Through each point on $I$ pass two characteristic curves with slopes $1/\tau_+$ and $1/\tau_-$ . Let $Q$ be one of the two points of inter-

section of the four characteristic curves through two adjacent




points $P_1$ and $P_2$ on I such that Q is on the side wherein we want to find the solution $x, y, u, v, \psi$ . We can solve for $x(Q)$ , $y(Q)$ from the finite difference equations

$$
\begin{aligned}
&x(Q) - x(P_1) = \tau_+(x(P_1), y(P_1), u(P_1), v(P_1), \psi(P_1))[y(Q)-y(P_1)] , \\
(6.2)\quad & \\
&x(Q) - x(P_2) = \tau_-(x(P_2), y(P_2), u(P_2), v(P_2), \psi(P_2))[y(Q)-y(P_2)] .
\end{aligned}
$$

Next, we can obtain $u(Q)$ and $v(Q)$ from

$$
\begin{aligned}
&A_1(P_1)[u(Q)-u(P_1)] + B_1(P_1)[v(Q)-v(P_1)] = C_1(P_1, Q) , \\
(6.3)\quad & \\
&A_2(P_2)[u(Q)-u(P_2)] + B_2(P_2)[v(Q)-v(P_2)] = C_2(P_2, Q) ,
\end{aligned}
$$

where $x, y, u, v$ at $P_1$ and $P_2$ plus the newly computed values of $x, y$ at Q are used to calculate the coefficients $A_i, B_i, C_i$ . Finally for $\psi(Q)$ , we have

$$
(6.4)\qquad \psi(Q) - \psi(P_1) = C_3(P_1, Q) .
$$

Thus far, this set of difference equations has a truncation error of first order because we have effectively approximated the derivatives by forward differences. We can obtain an error of second order if we use two iterations to compute $x,y,u,v,\psi$ at $Q$ in the following manner. First the equations (6.2-4) are solved for $\hat{x},\hat{y},\hat{u},\hat{v},\hat{\psi}$ at $Q$. Then using the averages $\frac{1}{2}[x(P_1) + \hat{x}(Q)]$, $\frac{1}{2}[x(P_2)+\hat{x}(Q)]$, etc. instead of $x(P_1)$, $x(P_2)$, etc. in the calculation of the coefficients $\tau_{\pm}$, $A_i,B_i,C_i$, we use equations (6.2-4) a second time to obtain $x,y,u,v,\psi$ at $Q$. The combined effect of the two iterations is equivalent to approximating the differential equations by centered differences, thus giving us a truncation error of second order.

This scheme may be used to solve the system (4.1) both in the subsonic and supersonic regions, the only difference being that in the subsonic region, all the quantities are complex variables of $r$ and $t$. The reflection principle may be taken advantage of, however, because $x,y,u,v,\psi$ are real on the $t$-axis, and calculations need to be carried out only in one quadrant of the $r,t$-plane.

To obtain a difference scheme with second order accuracy for the first order system (5.4), we need to do a little more work. Let us rewrite (5.4) as the nonlinear system

$$\begin{aligned} U_\zeta &= A(\xi,\eta,\zeta,U(\xi,\eta,\zeta))U_\xi + B(\xi,\eta,\zeta,U(\xi,\eta,\zeta))U_\eta \\ &\quad + C(\xi,\eta,\zeta,U(\xi,\eta,\zeta)) \end{aligned} \tag{6.5}$$

with the prescribed Cauchy data $U(\xi,\eta,0)$ . The most obvious scheme to use is that involving centered differences. Superimpose a rectangular grid in the $\xi,\eta,\zeta$-domain such that mesh sizes along the $\xi$, $\eta$ and $\zeta$ axes are $h_1$, $h_2$ and $k$, respectively. Then

$$
\begin{aligned}
U(\xi,\eta,\zeta+k) &= U(\xi,\eta,\zeta-k) + \frac{k}{h_1} A(\xi,\eta,\zeta,U)[U(\xi+h_1,\eta,\zeta)-U(\xi-h_1,\eta,\zeta)] \\
&\quad + \frac{k}{h_2} B(\xi,\eta,\zeta,U)[U(\xi,\eta+h_2,\zeta) - U(\xi,\eta-h_2,\zeta)] \\
&\quad + 2k\, C(\xi,\eta,\zeta,U) + O(h^2 k) \quad .
\end{aligned}
\tag{6.6}
$$

In this form, we need the values of $U$ at six points to determine $U$ at a new point. To decrease the amount of computation, we note that instead of using $U(\xi,\eta,\zeta)$ in the computation of the matrices $A$, $B$, and $C$, we can use the average

$$
\begin{aligned}
\overline{U} &= \frac{1}{4} [U(\xi+h_1,\eta,\zeta) + U(\xi-h_1,\eta,\zeta) \\
&\quad + U(\xi,\eta+h_2,\zeta) + U(\xi,\eta-h_2,\zeta)] = U(\xi,\eta,\zeta) + O(h^2) \ .
\end{aligned}
\tag{6.7}
$$

This would introduce an error of order $h^2$ in the calculation of the matrices $A$, $B$, and $C$ because

$$
\begin{aligned}
A(\xi,\eta,\zeta,U + O(h^2)) &= A(\xi,\eta,\zeta,U) + O(h^2)\, A_U(\xi,\eta,\zeta,U) \\
&= A(\xi,\eta,\zeta,U) + O(h^2) \ ,
\end{aligned}
$$

if we assume that $|A_U| < K$ in our domain of solution. This error of order $h^2$ in the matrices A, B, and C does not increase our truncation error of order $k^3$ for $U(\xi,\eta,\zeta+k)$.

Thus our difference scheme now has the form

$$
\begin{aligned}
U(\xi,\eta,\zeta+k) &= U(\xi,\eta,\zeta-k) \\
&\quad + \frac{k}{h_1} A(\xi,\eta,\zeta,\overline{U})[U(\xi+h_1,\eta,\zeta) - U(\xi-h_1,\eta,\zeta)] \\
&\quad + \frac{k}{h_2} B(\xi,\eta,\zeta,\overline{U})[U(\xi,\eta+h_2,\zeta) - U(\xi,\eta-h_2,\zeta)] \\
&\quad + 2k\, C(\xi,\eta,\zeta,\overline{U}) + O(k^3) \quad ,
\end{aligned} \tag{6.8}
$$

where $\overline{U}$ is defined in (6.7). To determine U at $(\xi,\eta,\zeta+k)$, we need to know it only at the alternate points $(\xi+h_1,\eta,\zeta)$, $(\xi-h_1,\eta,\zeta)$, $(\xi,\eta+h_2,\zeta)$, $(\xi,\eta-h_2,\zeta)$ and $(\xi,\eta,\zeta-k)$.

However, we note that the values of U at two levels of $\zeta$ are needed in order to compute it at a third level. As our problem is stated, this is impossible since we are given U only at the initial $\zeta$-level. At the start we need an auxiliary scheme which will produce a second $\zeta$-level of U values from the initial $\zeta$-level. In order to be compatible with our main difference scheme, this auxiliary scheme must give at least second order accuracy. Although we do not have to worry about the stability of the scheme since we are to use it only once, we have to be concerned with the accuracy of the results on the second $\zeta$-level. The success of the stability analysis on the

main difference scheme (6.8), that we shall make later, depends on how close to the truth we are when we assume that we have two levels of analytic initial data. Since the difference scheme is an approximation to the differential equations, the solution that we shall obtain at the second $\zeta$-level differs from the true solution by some error. This error will be of an oscillatory nature since our equations are of hyperbolic type. Hence if the mesh width in the $\zeta$-direction is reduced, this error function will be damped out.

We shall now proceed with the derivation of the auxiliary difference scheme. We expect beforehand that this scheme will involve more than the five points required by (6.8) in order to compensate for the lack of a second $\zeta$-level. First note that

$$U(\xi,\eta,k) = U(\xi,\eta,0) + k\, U_\zeta(\xi,\eta,\tfrac{k}{2}) + O(k^3) \tag{6.9}$$

This implies that we need to know $U_\zeta(\xi,\eta,\frac{k}{2})$ up to $O(k^2)$. Now from (6.5)

$$\begin{aligned} U_\zeta(\xi,\eta,\tfrac{k}{2}) &= A(\xi,\eta,\tfrac{k}{2},U)\, U_\xi(\xi,\eta,\tfrac{k}{2}) \\ &\quad + B(\xi,\eta,\tfrac{k}{2},U)\, U_\eta(\xi,\eta,\tfrac{k}{2}) + C(\xi,\eta,\tfrac{k}{2},U) \; ; \end{aligned} \tag{6.10}$$

hence we see that we actually need to know $U$, $U_\xi$, $U_\eta$ up to $O(k^2)$ at the intermediate level $\zeta = \frac{k}{2}$. Expanding $U$, $U_\xi$, $U_\eta$

and using (6.5) we obtain

$$U(\xi,\eta,\tfrac{k}{2}) = U(\xi,\eta,0) + \frac{k}{2} U_\zeta(\xi,\eta,0) + O(k^2)$$

(6.11) $$= U(\xi,\eta,0) + \frac{k}{2} [A(\xi,\eta,0,U)U_\xi(\xi,\eta,0)$$

$$+ B(\xi,\eta,0,U)U_\eta(\xi,\eta,0) + C(\xi,\eta,0,U)] + O(k^2) ,$$

and

$$U_\xi(\xi,\eta,\tfrac{k}{2}) = U_\xi(\xi,\eta,0) + \frac{k}{2} U_{\xi\zeta}(\xi,\eta,0) + O(k^2)$$

$$= U_\xi(\xi,\eta,0) + \frac{k}{2}\frac{1}{2h_1} [U_\zeta(\xi+h_1,\eta,0) - U_\zeta(\xi-h_1,\eta,0)] + O(h^2)$$

$$= U_\xi(\xi,\eta,0) + \frac{k}{4h_1} [A(\xi+h_1,\eta,0,U)\ U_\xi(\xi+h_1,\eta,0)$$

(6.12) $$+ B(\xi+h_1,\eta,0,U)\ U_\eta(\xi+h_1,\eta,0) + C(\xi+h_1,\eta,0,U)$$

$$- A(\xi-h_1,\eta,0,U)\ U_\xi(\xi-h_1,\eta,0)$$

$$- B(\xi-h_1,\eta,0,U)\ U_\eta(\xi-h_1,\eta,0) - C(\xi-h_1,\eta,0,U)]$$

$$+ O(k^2).$$

A similar expansion holds for $U_\eta(\xi,\eta,\frac{k}{2})$ . The expressions we have obtained for $U$, $U_\xi$, and $U_\eta$ involve only the space derivatives of $U$ on the initial level and we shall use centered dif-

erences to approximate them. In computing the coefficient matrices A, B, and C at $(\xi,\ \eta,\ \frac{k}{2},\ U(\xi,\eta,\frac{k}{2}))$ in equation (6.10), we shall use the approximation (6.11) for $U(\xi,\eta,\frac{k}{2})$.
Altogether, our auxiliary difference scheme is

$$
\begin{aligned}
U(\xi,\eta,k) &= U(\xi,\eta) + k\Big\{A(\xi,\eta,\tfrac{k}{2},\overline{U})E_1 \\
&\quad + B(\xi,\eta,\tfrac{k}{2},\overline{U})E_2 + C(\xi,\eta,\tfrac{k}{2},\overline{U})\Big\} + O(k^3)\ ,
\end{aligned}
\tag{6.13}
$$

where

$$\overline{U} = U(\xi,\eta) + F(\xi,\eta)\ ,$$

$$
\begin{aligned}
E_1 &= \frac{1}{2h_1}\left[U(\xi+h_1,\eta) - U(\xi-h_1,\eta)\right] \\
&\quad + \frac{k}{4h_1}\left[F(\xi+h_1,\eta) - F(\xi-h_1,\eta)\right]\ ,
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= \frac{1}{2h_2}\left[U(\xi,\eta+h_2) - U(\xi,\eta-h_2)\right] \\
&\quad + \frac{k}{4h_2}\left[F(\xi,\eta+h_2) - F(\xi,\eta-h_2)\right]\ ,
\end{aligned}
$$

and where

$$
\begin{aligned}
F(\xi,\eta) &= \frac{1}{2h_1}\Big\{A(\xi,\eta,0,U)\left[U(\xi+h_1,\eta) - U(\xi-h_1,\eta)\right]\Big\} \\
&\quad + \frac{1}{2h_2}\Big\{B(\xi,\eta,0,U)\left[U(\xi,\eta+h_2) - U(\xi,\eta-h_2)\right]\Big\} \\
&\quad + C(\xi,\eta,0,U)\ .
\end{aligned}
$$

Thus we have a difference scheme which approximates the differential equations to second order accuracy. Up to this point we have employed quite laborious means to get a good solution at the second level. Next we have to worry about what happens to our solution as the calculation is carried out from $\zeta = 0$ to $\zeta = T$. The main point of a stability analysis is to examine the extent to which any component of an initial function can be amplified by iterations of the numerical procedure.

For the purpose of this analysis, we consider A, B, and C as constant matrices. We then write the difference scheme (6.8) as

$$
\begin{aligned}
U(\xi,\eta,\zeta+k) &= U(\xi,\eta,\zeta-k) + \frac{k}{h_1} A[U(\xi-h_1,\eta,\zeta) - U(\xi-h_1,\eta,\zeta)] \\
&\quad + \frac{k}{h_2} B[U(\xi,\eta+h_2,\zeta) - U(\xi,\eta-h_2,\zeta)] + 2kC \,.
\end{aligned}
\tag{6.14}
$$

In its present form, we have a three-level difference scheme. For convenience let us reduce it to a two-level system in the following manner:

$$
\begin{aligned}
U(\xi,\eta,\zeta+k) &= V(\xi,\eta,\zeta) + A\frac{k}{h_1}[U(\xi+h_1,\eta,\zeta) - U(\xi-h_1,\eta,\zeta)] \\
&\quad + B\frac{k}{h_2}[U(\xi,\eta+h_2,\zeta) - U(\xi,\eta-h_2,\zeta)] + 2kC, \\
V(\xi,\eta,\zeta+k) &= U(\xi,\eta,\zeta) \,.
\end{aligned}
\tag{6.15}
$$

In essence we have a system of the form

$$W^{n+1}(\xi,\eta) = R(k)\, W^{n}(\xi,\eta) + S(k) \ . \tag{6.16}$$

As our calculations go from $\zeta = 0$ to $\zeta = T$ , we are in effect applying the sequence of operators $R^{n}(k)$ , $0 \leq nk \leq T$ , on the initial data $W^{o}(x,y)$ . As defined in [12], the operator $R(k)$ is said to be stable if the set of operators $R^{n}(k)$ , $n = 1,\dots,T/k$ is uniformly bounded for $k$ small enough.

Following the procedure described in [12], we assume that (6.16) is solvable for periodic $W(\xi,\eta)$ , which is uniquely determined by the equations and periodicity conditions. Then we can use the Fourier series representation of the solution to obtain a stability criterion. We write $W(\xi,\eta)$ as

$$W(\xi,\eta) = \sum_{r_1,r_2=1}^{\infty} Z(\nu_1,\nu_2)\, e^{i(\nu_1\xi+\nu_2\eta)} \ ,$$

where $\nu_1 = 2\pi\, r_1/L_1$ and $\nu_2 = 2\pi\, r_2/L_2$ are the frequencies corresponding to the periods $L_1$ and $L_2$ of $W(\xi,\eta)$ in the $\xi$- and $\eta$- directions, respectively, and $Z(\nu_1,\nu_2)$ is the amplitude of the harmonics of the fundamental vibration $e^{2\pi i(\xi/L_1+\eta/L_2)}$ . The equations (6.16) then become

$$Z^{n+1}(\nu_1,\nu_2) = \begin{pmatrix} 2i\,[\frac{k}{h_1} A \sin \nu_1 h_1 + \frac{k}{h_2} B \sin \nu_2 h_2]\; I \\ I \qquad\qquad\qquad\qquad 0 \end{pmatrix} Z^n(\nu_1,\nu_2) \tag{6.17}$$

$$+ e^{-i(\nu_1 x+\nu_2 y)} \begin{pmatrix} 2kC \\ 0 \end{pmatrix} ,$$

where the $6 \times 6$ coefficient matrix

$$G(k,\nu_1,\nu_2) = \begin{pmatrix} 2i[A \frac{k}{h_1} \sin \nu_1 h_1 + B \frac{k}{h_2} \sin \nu_2 h_2] & I \\ I & 0 \end{pmatrix} \tag{6.18}$$

is called the amplification matrix.

We shall use the following definition of matrix norm:

$$\| G \| = \max_{|w|\neq 0} \frac{|Gw|}{|w|} = \max_{|w|=1} |Gw| ; \tag{6.19}$$

where $|w| = \sqrt{w_1^2 + \ldots + w_m^2}$ . Now the matrix $G$ can be partitioned in the form

$$G = \begin{pmatrix} G_1 & I \\ I & 0 \end{pmatrix} ,$$

where $G_1$ is the $3 \times 3$ matrix

$$G_1 = 2i[A \frac{k}{h_1} \sin \nu_1 h_1 + B \frac{k}{h_2} \sin \nu_2 h_2] ,$$

and $I$ is a $3 \times 3$ identity matrix. By construction, the

matrices $A$ and $B$ are Hermitian, hence their eigenvalues are all real. Let $\lambda_A$ and $\lambda_B$ denote the maxima of the eigenvalues of $A$ and $B$. Then the amplification matrix $G$ is majorized by the $6 \times 6$ matrix

$$(6.21) \qquad \Gamma = \begin{pmatrix} 2i\left[\lambda_A \dfrac{k}{h_1} \sin \nu_1 h_1 + \lambda_B \dfrac{k}{h_2} \sin \nu_2 h_2\right] I & I \\ I & 0 \end{pmatrix},$$

and we need only study the eigenvalues of $\Gamma$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_6$ be the eigenvalues of the matrix $\Gamma$. The maximum of $|\lambda_i|$, $i = 1, \ldots, 6$ is called the spectral radius $\sigma$ of $\Gamma$. Clearly $\sigma \leq \|\Gamma\|$ because the maximum value of $|\Gamma w|/|w|$ is not less than the value obtained by substituting for $w$ an eigenvector of $\Gamma$ corresponding to its largest eigenvalue. Hence also $\sigma^n \leq \|\Gamma^n\|$ and a necessary condition for stability is that there exists a constant $K$ such that $\sigma^n(k, \nu_1, \nu_2) \leq K$ for $0 < k < \tau$, $0 \leq nk \leq T = Nk$. Without loss of generality assume that $1 \leq K$. Then we have that

$$\sigma(k, \nu_1, \nu_2) \leq K^{1/N} \quad K^{k/T} .$$

Now for $k$ in the interval $0 \leq k \leq \tau$, with small $\tau$

$$K^{k/T} \leq 1 + K_1 k , \qquad K_1 \text{ some constant,}$$

hence a necessary condition for stability is that the eigenvalues of $\Gamma$ must satisfy

$$|\lambda_i| \leq 1 + O(k) \quad \text{for} \quad 0 \leq k \leq \tau \ , \ i = 1,\ldots,6 \ .$$

The matrix $\Gamma$ has the triple pair of eigenvalues of the form

(6.22) $$\lambda = \frac{1}{2} \left( i\omega \pm \sqrt{4-\omega^2} \right) \ ,$$

where

$$\omega = 2\left(\lambda_A \frac{k}{h_1} \sin \nu_1 h_1 + \lambda_B \frac{k}{h_2} \sin \nu_2 h_2\right) \ .$$

In order that $|\lambda| = 1$ , $\omega$ must satisfy the inequality $|\omega| \leq 2$ . Thus a necessary condition for stability is that

(6.23) $$\left|\lambda_A \frac{k}{h_1} \sin \nu_1 h_1 + \lambda_B \frac{k}{h_2} \sin \nu_2 h_2\right| \leq \left|\lambda_A \frac{k}{h_1} + \lambda_B \frac{k}{h_2}\right| \leq 1 \ .$$

It can be shown (see [12]) that condition (6.23) becomes sufficient for stability if the equality sign is omitted. In that case, the determinant of the eigenvectors of $\Gamma$

$$-\frac{1}{2} \left\{ (4-\omega^2) - i\omega \sqrt{4-\omega^2} \right\}$$

is bounded away from zero.

Since in our problem the matrices A and B are not

constant, but depend on the solution, the quantities $\lambda_A$ and $\lambda_B$ were computed at every point of the grid during the actual calculation so that the stability of the run could be checked.

The computation procedures for the coefficients in both methods are essentially the same. Given $u, v, \psi$ at a point, the various quantities $A(\psi)$, $c^2$, $\rho$, $p$, $A'(\psi)/A(\psi)$ and $Q$ are calculated in the following order:

$$N_o^2 = \frac{U}{1+f'^2(\frac{\psi}{U})} \quad , \qquad\qquad U = \sqrt{\gamma} \quad M \quad ,$$

$$A(\psi) = N_o^{-2\gamma} \; [(1-\mu^2)\; N_o^2-\mu^2][\mu^2 N_o^2 + (1+\mu^2)]^\gamma \; , \qquad \mu^2 = \frac{\gamma-1}{\gamma+1} \quad ,$$

$$\rho^{\gamma-1} = (H - \frac{u^2+v^2}{2}) \quad \frac{\gamma-1}{\gamma A(\psi)} \quad ,$$

$$c^2 = \gamma A(\psi) \; \rho^{\gamma-1} \quad ,$$

$$\rho = \exp\left(\frac{1}{\gamma-1} \quad \log \rho^{\gamma-1}\right) \quad ,$$

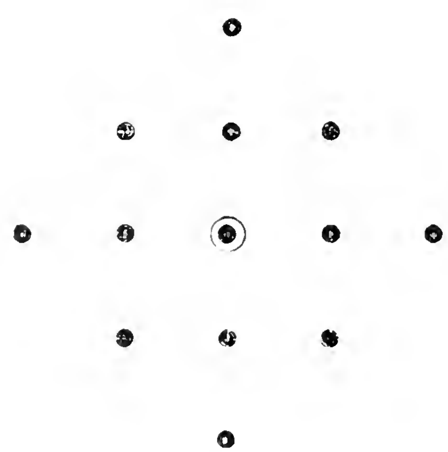
$$p = A(\psi) \; \rho^{\gamma-1} \; \rho \quad ,$$

$$\frac{A'(\psi)}{A(\psi)} = -2 \; \frac{f'(\frac{\psi}{U})f''(\frac{\psi}{U})}{U[1+f'^2(\frac{\psi}{U})]} \quad \frac{(1-\mu^2)N_o^2[\mu^2 N_o^2+(1+\mu^2)]-\gamma(1+\mu^2)[(1-\mu^2)N_o^2-\mu^2]}{[(1-\mu^2)N_o^2-\mu^2][\mu^2 N_o^2+(1+\mu^2)]} \quad ,$$

and finally

$$Q = \frac{c^2}{\gamma(\gamma-1)} \quad \frac{A'(\psi)}{A(\psi)} \quad \rho \quad .$$

To solve the system (5.4), we set up a triangular grid in the $\xi,\eta$-plane with its base on the $\xi$-axis. We are able to confine our calculations to the upper half-plane because of the symmetry properties acquired by the functions $u, v, p$, and $\rho$ under analytic continuation. By the reflection principle, any function $f$ which is real-valued on the $\xi$-axis may be analytically continued into the complex plane by the rule $f(\xi - i\eta) = \overline{f(\xi+i\eta)}$. We compute initial values at every grid point, then use the auxiliary scheme to obtain a second $\zeta$-level of solutions. Since the latter is a 13-point scheme using the points indicated in the diagram, the triangular grid is decreased by two boundary rows at the second $\zeta$-level. However, from then on, we use the 5-point main difference scheme, which shrinks the grid by only one boundary row at each $\zeta$-level. We have, in the end a tetrahedral domain of answers in the $\xi,\eta,\zeta$-space, of which only its triangular section in the real $\zeta,\xi$-plane is of interest to us.



## 7. Numerical Results

In the preceding sections we derived a formulation of the inverse detached shock problem in terms of the unknowns $u, v$, and $\psi$. We found that the equations of motion, treated either as a first-order symmetric hyperbolic system (5.4) or as a system in characteristic normal form (6.1), plus initial conditions on the shock constituted a well-posed Cauchy problem in the complex domain. We then devised difference schemes of second order accuracy to solve the Cauchy problem in its two versions.

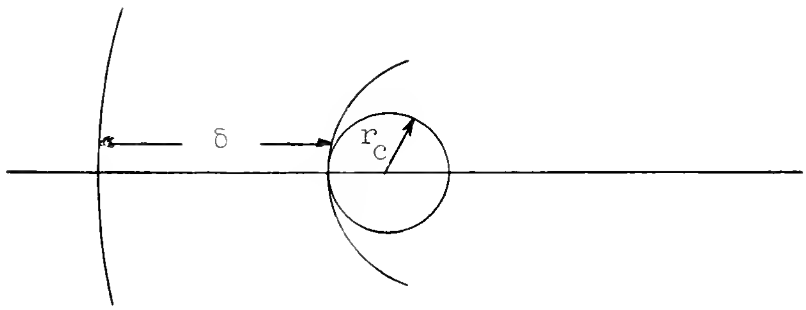
Our next task is to check the feasibility of the method in practice. The stability condition (6.23) was derived under the stringent assumption that the coefficient matrices were constant. In reality, our equations are nonlinear and we want to see whether the stability estimate we made is still valid. We also want to know whether the inverse problem can be solved for the flow quantities and the body in a reasonable amount of machine time. In addition, because of the nonlinearity of the equations, there is the question of how much of the body we can solve for in the large. Finally, although theory points to the three-dimensional system (5.4) as the better of the two versions, we want to see whether such a preference is justifiable in practice.

All versions discussed in Section 2 were programmed in the FORTRAN II language for the IBM 7094 at the NYU AEC Computing Center. The $u,v,p,\rho$- and $\psi,\psi_s,\psi_t$-versions served only as checks on the $u,v,\psi$-programs and no significant results were

obtained from them. For finding actual flows, only two programs were used: Program I, which applied the difference schemes (6.13) and (6.8) to the system (5.4) in the three-dimensional $\xi,\eta,\zeta$-domain, and Program II, which integrated the system (6.1) along characteristics by Massau's method (6.2-4). Program I uses 8729 machine locations and has sufficient storage space for a 59-iteration run. For more iterations, tapes or discs would have to be used for temporary storage. A listing of machine instructions for Program I is included in this paper. Program II has two forms: the first employs only real arithmetic and is intended for use in the supersonic region; the second employs complex arithmetic and is used in the subsonic region. Program II(real) takes up 5524 locations and has sufficient storage space for an 897-iteration run. Program II(complex) uses 6718 locations and can do a run of at most 857 iterations. In quoting the sizes of these programs, we are including in the count the storage space required by the utility subroutines used by the program.

We shall now discuss the results obtained for a flow at Mach 6 with the shock wave $x = 3\sqrt{1+y^2}$. In Figure 1, we exhibit that portion of the flow, body and sonic line which a single 45-iteration run with Program I gave. This run took 13.2 minutes on the IBM 7094 and the results were good to four significant figures. Table 1 indicates the convergence of the answers at a point in the subsonic region. Table 4 lists the coordinates of the interpolated body points, and the pressure, velocities

and Mach number at these points. The ratio of the detachment distance $\delta$ to twice the radius of curvature $r_c$ of the body at the vertex was found to be $\delta/2r_c = .217$ . This compares favorably with the data obtained by Kim [9], who made experimental studies of flows past cylindrical bodies. A second quantity which Kim considered was the ratio of twice the radius of curvature to the sum of the detachment distance and the radius of curvature. In our case, this quantity was $2r_c/\delta+r_c = .82$ .




In Figures 2 and 3, we exhibit the results obtained for the same flow using the method of characteristics. Heavy lines mark that portion of the sonic line and the body solved for by Program II. In Figure 3, we display the section of the transonic region which lies outside the range of influence of the shock wave. Figure 3 also includes the continuation of the sonic line in front of the body obtained by integrating the equations (6.1) along characteristics in the direction away from the body. Finally, notice in Figures 2 and 3 the three-cornered path of answers obtained by using a similarly-shaped initial curve in the complex $t,r$-plane. This illustrates the comment made in Section 4 on the possibility of using an initial

path in the complex domain which is not analytic.

In the subsonic region, 60 steps were needed to get to the sonic line behind the body with results good to five significant figures. Such a run took 1.8 minutes and Table 2 shows the convergence of the answers at a typical point. Note that to locate the body and the sonic line adequately, four or five values of $s$ had to be used. In the supersonic region, 120 iterations were needed to get body points with five digit accuracy. Such a run took 4.2 minutes and Table 3 illustrates the convergence of the answers.

As a check on the accuracy of the calculations, the various versions of the program were used to compute the flow quantities at particular points. The answers obtained agreed with one another to five significant figures as Tables 5, 6 and 7 indicate.

We should like to note that the run exhibited in Figure 1 involves an additional transformation of the $x,\zeta$-domain, namely

$$x' = x + m\zeta \ , \quad \zeta' = \sqrt{m^2 + 1}\ \ \zeta \ , \qquad m \text{ constant} .$$

This transformation rotates the integration path and allows us to approach the body in a direction which more closely approximates the $\tau_+$ characteristic direction. The best value to use for $m$ depends on the inclination of the $\tau_+$-characteristics in the physical plane with respect to the $x$-axis. For the particular run in Figure 1, we used $m = 1/2$ . This rotation

is most needed in the supersonic region where the $\tau_+$-characteristics slant toward the x-axis.

Aside from the particular flow discussed above, we considered various other shock shapes and flow speeds. As to the shock shape $x = f(y) = f(\xi)$, there are some limitations on our choice. Foremost is the necessity to choose elementary and analytic expressions for $f(\xi)$ in order that analytic continuation of the initial conditions can be performed in closed form. A second limitation arises if we consider the formulas (3.9). The term $1 + f'^2(\xi)$ occurs in the denominator and hence we have to worry about the zeros of this expression. For example, consider the hyperbola $x = \sqrt{M^2-1}\ \sqrt{1+y^2}$, whose asymptotes are the characteristics of the incident flow. After analytic continuation is performed on $\xi = y$, we have that

$$f'(\varkappa) = \sqrt{M^2-1}\ (1+\varkappa^2)^{-1/2}\varkappa ,$$

where $\varkappa = \xi + i\eta$. To avoid the points $\varkappa = \pm i/M$ at which the expression $1 + f'^2(\varkappa) = 0$, we have to limit the extent of our initial domain in the $\eta$-direction, which, in turn, controls the size of the domain of solution through the stability condition (6.23). The principal difficulty in practice is to ensure that the allowable domain of solution will not fall short of the body. Hence the choice of a shock wave $f(\varkappa)$ is also influenced by the locations of the roots of $1 + f'^2(\varkappa) = 0$. It is desirable to have them as sparse in number and as far from the origin as possible.

While experimenting with different shock shapes, we came upon a third factor that has to be taken into account when choosing a shock. With the use of Program II, it was found that for $M = 6$ the flow corresponding to the shock wave $x = \sqrt{M^2-1} \; \sqrt{1+y^2}$ contained an envelope of characteristics in the supersonic region between the shock and the body. In terms of the physical flow, this envelope could be interpreted as a secondary shock occurring in the flow, a phenomenon which is not unusual. Although no difficulty was encountered in getting to the body and the sonic line behind the body using Program II (complex) in the subsonic region, we could not reach the body in the supersonic region with Program II (real) because of such a singularity in the flow. Program I yielded even less information than was obtained from Program II (real) because the extent of its region of solution was limited by the envelope of characteristics in the supersonic region. Thus, we have to conclude that in order for Program I to give satisfactory answers, the flow behind the shock wave must be continuous and free of any secondary shocks. For the case we considered at Mach 6, we were then led to try the hyperbola $x = 3 \sqrt{1+y^2}$ .

Even when Program I fails, however, we are still able to get valuable information about the flow and the body by using Program II. Using the complex arithmetic version, the portion

of the body in the subsonic region can be solved for, while the real version used in the supersonic region gives us an indication of singularities in the flow. For flows with small Mach numbers, e.g. $M = 1.2$, we have not yet found a shock shape which does not have accompanying secondary shocks in the supersonic region. Finding a shock shape suitable for Program I is a harder job for small Mach numbers because the detachment distance $\delta$ increases as $M$ decreases. This means that we have to solve the nonlinear problem in an even larger domain. Moreover, from the physical point of view, secondary shocks are expected when $M$ is small.

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## TABLES

The following tables contain results from the accuracy and convergence tests performed for a flow at Mach 6, $\gamma = 1.4$ with the shock wave $x = 3\sqrt{1+y^2}$ .

### Table 1

Convergence at the point $x = 3.0696592$ , $y = .041458976$ using the three-dimensional method.

| No. of Iterations | $h_1$ | $h_2$ | k | p |
|---|---|---|---|---|
| 10 | .05 | .025 | .0125 | 43.206604 |
| 20 | .025 | .0125 | .00625 | 43.259683 |
| 40 | .0125 | .00625 | .003125 | 43.266751 |

### Table 2

Convergence at a point using the method of characteristics in the subsonic region.

| No. of Iterations | $\Delta r$ | x | y | p |
|---|---|---|---|---|
| 15 | .0112 | 3.0970604 | .051564741 | 35.568517 |
| 30 | .0056 | 3.0970272 | .051679900 | 35.562432 |
| 60 | .0028 | 3.0970185 | .051709075 | 35.561677 |

<u>Table 3</u>

Convergence at a point using the method of characteristics in the supersonic region.

| No. of Iterations | $\Delta y$ | x | y | p |
|---|---|---|---|---|
| 30 | .008 | 3.2109012 | .14448365 | 13.024765 |
| 60 | .004 | 3.2108428 | .14457851 | 13.022444 |
| 120 | .002 | 3.2108301 | .14460678 | 13.022385 |

<u>Table 4</u>

Interpolated flow quantities on the body obtained from three-dimensional scheme.

| x | y | $p/p_{max}$ | M | $\tan^{-1}\frac{u}{v}$ |
|---|---|---|---|---|
| 3.0657 | 0. | 1. | 0. | 0. |
| 3.0666 | .016902 | .98746 | .14224 | .12363 |
| 3.0687 | .028756 | .96176 | .24144 | .19968 |
| 3.0716 | .040378 | .92476 | .33969 | .28259 |
| 3.0755 | .051729 | .87812 | .43773 | .35619 |
| 3.0803 | .062780 | .82355 | .53640 | .44625 |
| 3.0860 | .073525 | .76296 | .63603 | .52680 |
| 3.0926 | .083929 | .69838 | .73680 | .60672 |
| 3.1002 | .093971 | .63189 | .83865 | .68614 |
| 3.1088 | .10361 | .56479 | .94329 | .76536 |
| 3.1183 | .11283 | .50107 | 1.0462 | .84259 |
| 3.1289 | .12158 | .44406 | 1.1438 | .91507 |
| 3.1404 | .12988 | .39739 | 1.2295 | .97883 |

Table 5

Results at the point $x = 3.0068638$ , $y = .024$ after five iterations using various three-dimensional formulations of the flow problem.

| | $p$ | $M^2$ |
|---|---|---|
| $u,v,p,\rho$-version | 42.385768 | .15766106 |
| $\psi,\psi_s,\psi_t$-version | 42.385007 | .15757674 |
| $u,v,\psi$-version | 42.384973 | .15757716 |

Table 6

Results at the point $x = 3.0591460$ , $y = .072465631$ using two forms of the $u,v,\psi$-version in the subsonic region.

| | $p$ | $M^2$ | No. of Iterations |
|---|---|---|---|
| 3-dimensional method | 39.828291 | .25751588 | 64 |
| method of characteristics | 39.825180 | .25760289 | 18 |

Table 7

Results at the point $x = 3.1388759$ , $y = .21003801$ using two forms of the $u,v,\psi$-version in the supersonic region.

| | $p$ | $M^2$ | No. of Iterations |
|---|---|---|---|
| 3-dimensional method | 21.027724 | 1.6820741 | 56 |
| method of characteristics | 21.022659 | 1.6825560 | 36 |

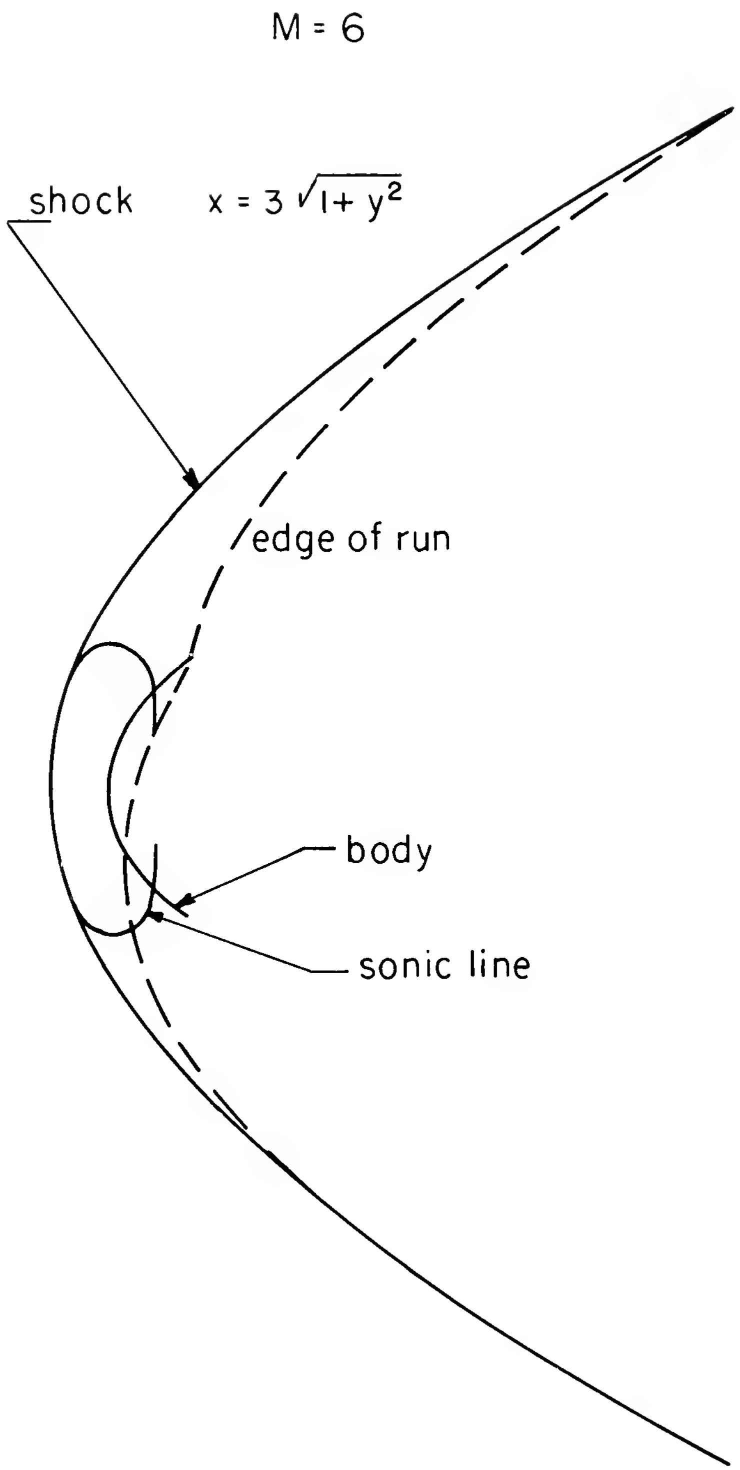


Figure 1

Calculation of the flow based on one run of the three-dimensional method.

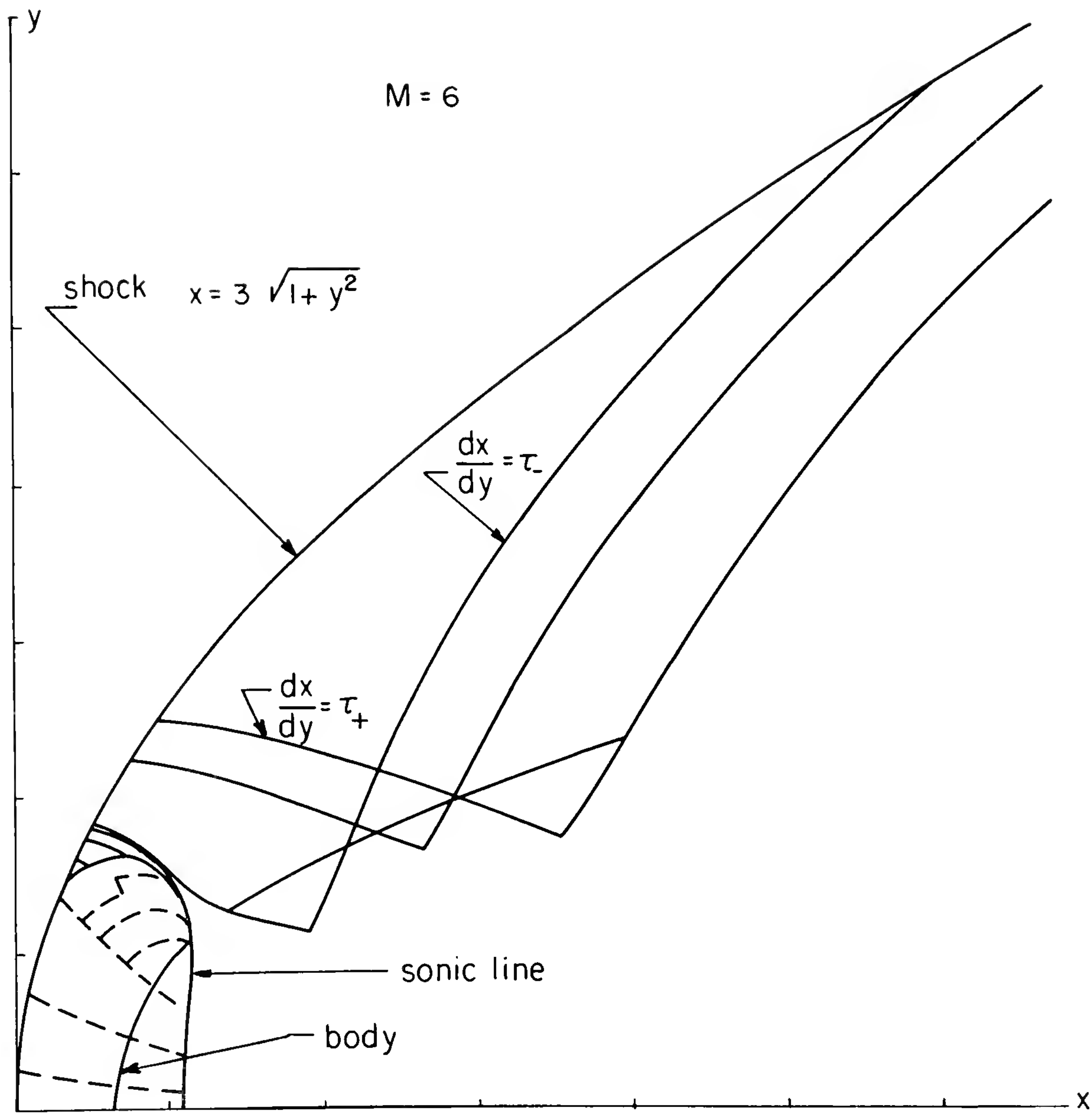


Figure 2

Portion of flow obtained using the method of characteristics. Characteristics are displayed in the supersonic region. The dotted curves in the subsonic region indicate the intersection of the x,y-plane with the solution surface in complex domain.

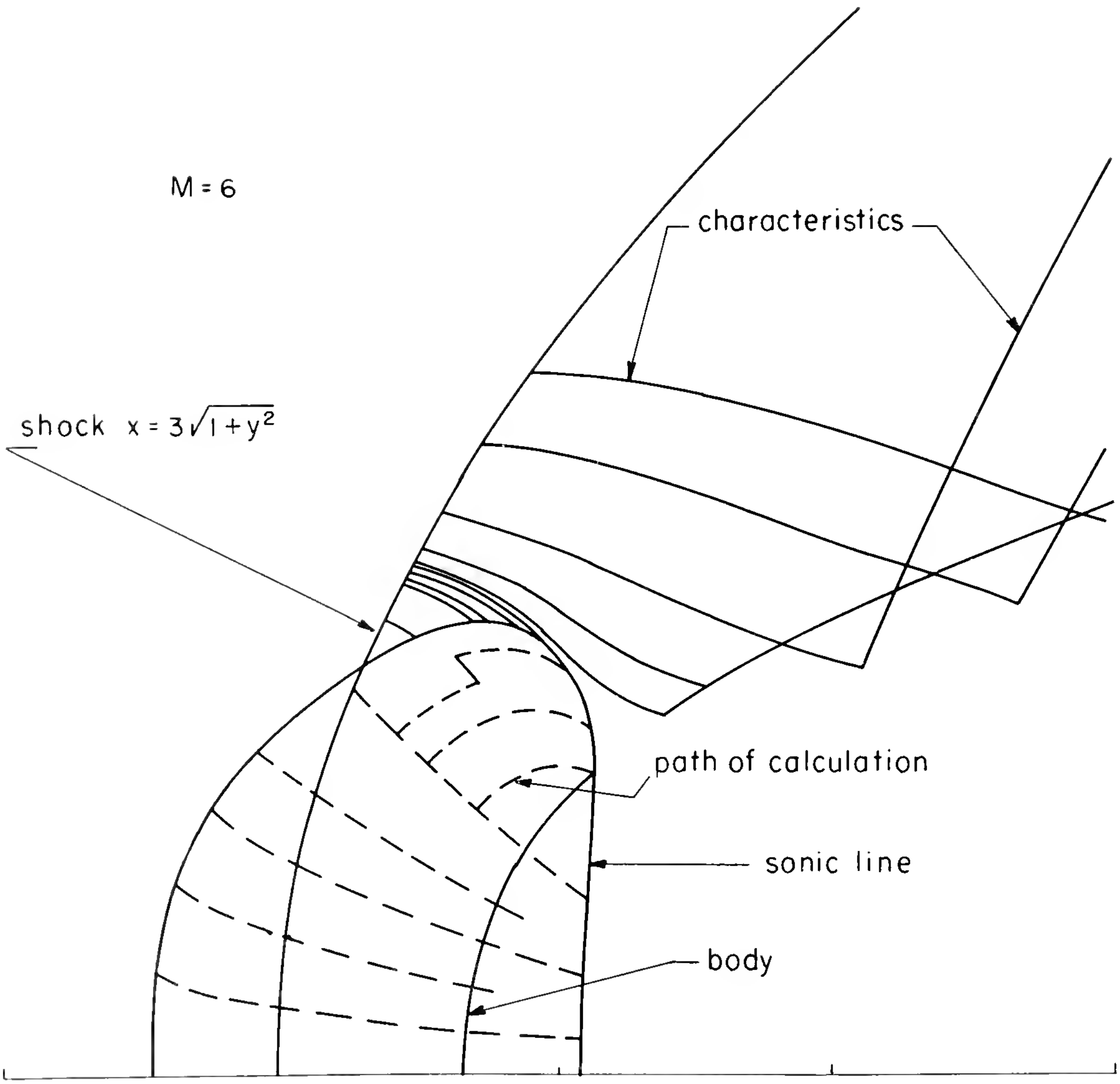


Figure 3

Enlargement of Figure 2 displaying the transonic region and featuring the continuation of the sonic line in front of the shock wave.

```
CSHØCKR     NUMERICAL CØMPUTATIØNS ØN DETACHED SHØCK WAVES
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,YØU,STARTX,PØW
      CØMMØN A,B,C,U,Q,R,S,T,SUM,WASH,SMACH,ITIME,SLØPE,GØSH
1     DIMENSIØN A(3,3),B(3,3),C(3),Q(3),R(3),S(3),T(3),SUM(3),U(61,61,3)
1     DIMENSIØN DELX(1),DELY(1),DELT(1),GAMMA(1),H(1),YØU(1),PØW(1)
1     DIMENSIØN SLØPE(1),WASH(121,2),GØSH(121,2)
      DIMENSIØN SMACH(121)
C
C        ITERAT = NØ. ØF ITERATIØNS DESIRED...MAY NØT BE GREATER THAN 59
C        IPRINT = 1  IF SPECTRAL RADII AT EVERY GRID PØINT IS DESIRED AS
C                        PART ØF ØUTPUT
C        IPRINT = 2  IF NØT INTERESTED IN PRINT ØUT ØF SPECTRAL RADII
C        GAMMA = RATIØ ØF SPECIFIC HEATS
C        PØW = MACH NUMBER --- CØRRESPØNDS TØ CAP M IN PAPER
C        DELX = MESH SIZE ALØNG KSI AXIS (= H1 IN PAPER)
C        DELY = MASH SIZE ALØNG ETA AXIS (= H2 IN PAPER)
C        DELT = MESH SIZE ALØNG ZETA AXIS (= K IN PAPER)
C        INITIAL CURVE ALØNG THE KSI AXIS RANGES FRØM STARTX TØ
C            ((((ITERAT+2)*2)-1)*DELX)+STARTX
C
C        THIS PRØGRAM USES TAPES 2 AND 3 FØR INTERMEDIATE STØRAGE
C
 3    READ INPUT TAPE 5,1,ITERAT,IPRINT,GAMMA,PØW,DELX,DELY,DELT,STARTX
      N=3
      NT=ITERAT-1
      NY=NT+3
      NX=(NY*2)-1
      NX2=(NX/2)+1
      NY2=NY-2
1     SLØPE=(0.5,0.)
      WRITE ØUTPUT TAPE 6,5
      WRITE ØUTPUT TAPE 6,4,PØW
1     YØU=PØW*SQRTF(GAMMA)
      WRITE ØUTPUT TAPE 6,2,NX,NY,STARTX,DELX,DELY,DELT,GAMMA,YØU,ITERAT
      CALL READY
      CALL ØUTPUT
      CALL AIM(IPRINT)
      CALL ØUTPUT
      CALL FIRE(IPRINT)
      GØ TØ (7,3),IPRINT
 7    WRITE ØUTPUT TAPE 6,6
 7    READ TAPE 3,I,J,IT,((A(IJ,KL),A(IJ,KL+3),B(IJ,KL),B(IJ,KL+3),IJ=1,
     X3),KL=1,3)
      CALL CEASE(A,BØUT)
      CALL CEASE(B,DEMIL)
      WRITE ØUTPUT TAPE 6,8,I,J,IT,BØUT,DEMIL
      IF(NT-IT) 3,3,7
 1    FØRMAT(2I3,2F4.1,4E13.8)
 2    FØRMAT(/33X19HGRID DIMENSIØNS AREI3,28H PØINTS IN THE KSI-DIRECTIØ
     XN/49X4HANDI3,28H PØINTS IN THE ETA-DIRECTIØN//33X37HLEFT END CØRNE
     XR ØF GRID IS LØCATED ATF12.6//33X4HH1 =1PE14.7,2X4HH2 =1PE14.7,2X3
     XHK =1PE14.7//33X7HGAMMA =0PF4.1,10X29HINITIAL HØRIZØNTAL VELØCITY
     X=1PE14.7///33X16HTHIS RUN WILL DØI3,12H ITERATIØNS.)
 4    FØRMAT(/33X57HGIVEN SHØCK WAVE IS THE HYPERBØLA  X = 3.*SQRTF(1 +
     XY**2)//33X20HIN THIS CASE MACH NUMBER M =F4.1)
 5    FØRMAT(1H1/61X10HSHØCKER VI)
 6    FØRMAT(1H1)
 8    FØRMAT(10X,3I5,1P2E20.8)
      END
```



```
C
CREADY      PREPARES INITIAL DATA USING SHØCK CØNDITIØNS
C           SEE EQNS (3.22) IN PAPER
      SUBRØUTINE READY
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,YØU,STARTX,PØW
      CØMMØN A,B,C,U,Q,R,S,T,SUM,WASH,SMACH,ITIME,SLØPE,GØSH
1     DIMENSIØN A(3,3),B(3,3),C(3),Q(3),R(3),S(3),T(3),SUM(3),U(61,61,3)
1     DIMENSIØN DELX(1),DELY(1),DELT(1),GAMMA(1),H(1),YØU(1),PØW(1)
1     DIMENSIØN SLØPE(1),WASH(121,2),GØSH(121,2)
      DIMENSIØN SMACH(121)
1     DIMENSIØN SS(1),HCAM(1)
      ITIME=0
1     GNU=(GAMMA-(1.,0.))/(GAMMA+(1.,0.))
1     H=((YØU*YØU)/(2.,0.))+(GAMMA/(GAMMA-(1.,0.)))
      DØ 1 J=1,NY
      NX1I=NX+1-J
      SS(2)=FLØATF(J-1)*DELY
      DØ 1 I=J,NX1I
      K=I-NX2
      L=NY+1-J
      IF(K) 2,2,3
 2    K=I
      L=J
 3    SS(1)=(FLØATF(I-1)*DELX)+STARTX
1     FPRIME=DERIV(SS)
1     CØS=(1.,0.)/SQRTF((1.,0.)+(FPRIME*FPRIME))
1     SIN=FPRIME*CØS
1     VECTAN=YØU*SIN
1     VECNØR=YØU*CØS
1     BØSTØN=VECNØR*VECNØR
1     CEESQ=(GNU*BØSTØN)+((1.,0.)*GNU
1     VECNØR=CEESQ/VECNØR
1     U(K,L,1)=(VECNØR*CØS)+(VECTAN*SIN)
1     U(K,L,2)=(VECTAN*CØS)-(VECNØR*SIN)
1     U(K,L,3)=SS*YØU
      IF(J-1) 1,4,1
14    WASH(I,2)=(((1.,0.)-GNU)*BØSTØN)-GNU
1     WASH(I,1)=U(K,L,3)
1     RHØ=BØSTØN/CEESQ
1     HCAM=(((U(K,L,1)*U(K,L,1))+(U(K,L,2)*U(K,L,2)))*RHØ)/(GAMMA*WASH(I
     X,2))
      SMACH(I)=SQRTF(HCAM(1))
1     GØSH(I,1)=U(K,L,1)
1     GØSH(I,2)=U(K,L,2)
 1    CØNTINUE
      RETURN
      END
C
CAIM      CØMPUTES FUNCTIØN AT SECØND ZETA-LEVEL USING AUXILIARY
C           DIFFERENCE SCHEME --- SEE EQN (6.13) IN PAPER
      SUBRØUTINE AIM(IPRINT)
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,YØU,STARTX,PØW
      CØMMØN A,B,C,U,Q,R,S,T,SUM,WASH,SMACH,ITIME,SLØPE,GØSH
1     DIMENSIØN A(3,3),B(3,3),C(3),Q(3),R(3),S(3),T(3),SUM(3),U(61,61,3)
1     DIMENSIØN DELX(1),DELY(1),DELT(1),GAMMA(1),H(1),YØU(1),PØW(1)
1     DIMENSIØN SLØPE(1),WASH(121,2),GØSH(121,2)
      DIMENSIØN SMACH(121)
1     DIMENSIØN U1(3),U2(3),TEMP(3)
      DIMENSIØN JAM(10),GAM(3)
```



```
      EQUIVALENCE (JAM(1),JAM1),(JAM(2),JAM2),(JAM(3),JAM3),(JAM(4),JAM4
     X),(JAM(5),JAM5),(JAM(6),JAM6),(JAM(7),JAM7),(JAM(8),JAM8),(JAM(9),
     XJAM9),(JAM(10),JAM10),(GAM(2),GAM2)
      REWIND 2
      REWIND 3
      ITIME=1
      NXM1=NX-1
      NY1=NY+1
1     C1=(2.,0.)*DELX
      C2=2.*DELY
1     CØNST1=DELT/((4.,0.)*DELX)
1     CØNS2=DELT/((4.,0.)*DELY)
      DØ 7 J=1,NY2
      JJ=NXM1-J
      KK=J+2
      DØ 7 I=KK,JJ,2
      CALL IVRLER(I,J,JAM,GAM)
      DØ 1 K=1,N
1     U1(K)=(U(JAM1,JAM6,K)-U(JAM2,JAM7,K))/C1
      U2(K)=(U(JAM3,JAM8,K)-U(JAM4,JAM9,K))/C2
1     U2(K+3)=(U(JAM3,JAM8,K+3)-(GAM2*U(JAM4,JAM9,K+3)))/C2
      CALL TEDIUM(I,J,SUM)
      CALL TEDIUM(I-2,J,S)
      DØ 5 K=1,N
15    T(K)=((SUM(K)-S(K))*CØNST1)+U1(K)
      CALL TEDIUM(I-1,J+1,SUM)
      CALL TEDIUM(I-1,J-1,S)
      DØ 4 K=1,N
14    SUM(K)=((SUM(K)-S(K))*CØNS2)+U2(K)
      CALL ABCHAR(I,J)
      CALL MAVE(A,T,Q,B,SUM,R)
      GØ TØ (10,11),IPRINT
10    WRITE TAPE 4,I,J,ITIME,((A(IJ,KL),A(IJ,KL+3),B(IJ,KL),B(IJ,KL+3),I
     XJ=1,3),KL=1,3)
11    DØ 6 K=1,N
16    TEMP(K)=U(JAM5,JAM10,K)+((Q(K)+R(K)+C(K))*DELT)
      WRITE TAPE 2,(TEMP(K),TEMP(K+3),K=1,N)
      IF(J-1) 7,9,7
 9    CALL THE(A,TEMP,I,J,0,2)
 7    CØNTINUE
      REWIND 2
      DØ 2 J=1,NY2
      JJ=NXM1-J
      KK=J+2
      DØ 2 I=KK,JJ,2
      III=I-NX2
      JJJ=NY1-J
      IF(III) 8,8,3
 8    III=I
      JJJ=J
 3    READ TAPE 2,(TEMP(K),TEMP(K+3),K=1,N)
      DØ 2 K=1,N
12    U(III,JJJ,K)=TEMP(K)
      RETURN
      END
C
CFIRE       USES MAIN DIFFERENCE SCHEME --- SEE EQN (6.3) IN PAPER
      SUBRØUTINE FIRE(IPRINT)
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,YØU,STARTX,PØW
```



```
      CØMMØN A,B,C,U,Q,R,S,T,SUM,WASH,SMACH,ITIME,SLØPE,GØSH
1     DIMENSIØN A(3,3),B(3,3),C(3),Q(3),R(3),S(3),T(3),SUM(3),U(61,61,3)
1     DIMENSIØN DELX(1),DELY(1),DELT(1),GAMMA(1),H(1),YØU(1),PØW(1)
1     DIMENSIØN SLØPE(1),WASH(121,2),GØSH(121,2)
      DIMENSIØN SMACH(121)
1     DIMENSIØN TEMP(3)
      DIMENSIØN JIN(10),GIN(3)
      EQUIVALENCE (JIN(1),JIN1),(JIN(2),JIN2),(JIN(3),JIN3),(JIN(4),JIN4
     X),(JIN(5),JIN5),(JIN(6),JIN6),(JIN(7),JIN7),(JIN(8),JIN8),(JIN(9),
     XJIN9),(JIN(10),JIN10),(GIN(2),GIN2)
1     C1=DELT/DELX
1     C2=DELT/DELY
1     C3=(2.,0.)*DELT
      DØ 4 KLM=1,NT
      ITIME=KLM+1
      NY2KLM=NY2-KLM
      DØ 3 J=1,NY2KLM
      JKLM=J+KLM
      JJ=NX-1-JKLM
      KK=2+JKLM
      DØ 3 I=KK,JJ,2
      CALL BØRDER(I,J,JIN,GIN)
      DØ 1 K=1,N
1     Q(K)=U(JIN1,JIN6,K)-U(JIN2,JIN7,K)
      R(K)=U(JIN3,JIN8,K)-U(JIN4,JIN9,K)
1     R(K+3)=U(JIN3,JIN8,K+3)-(GIN2*U(JIN4,JIN9,K+3))
      CALL ABC(I,J,1,1,1,-1,JIN,GIN)
      CALL MAVE(A,Q,S,B,R,U)
      GØ TØ (6,7),IPRINT
 6    WRITE TAPE 3,I,J,ITIME,((A(IJ,KL),A(IJ,KL+3),B(IJ,KL),B(IJ,KL+3),I
     XJ=1,3),KL=1,3)
 7    DØ 2 K=1,N
1     TEMP(K)=U(JIN5,JIN10,K)+C1*S(K)+C2*Q(K)+C3*C(K)
12    U(JIN5,JIN10,K)=TEMP(K)
      IF(J-1) 3,5,3
 5    CALL THE(A,TEMP,I,J,0,2)
 3    CØNTINUE
 4    CALL ØUTPUT
      REWIND 3
      RETURN
      END
C
CABCBAR     CØMPUTES MATRICES A,B,C AT (I1,I2,DELT/2) FØR SUBRTN AIM
      SUBRØUTINE ABCHAR(I1,I2)
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,YØU,STARTX,PØW
      CØMMØN A,B,C,U,Q,R,S,T,SUM,WASH,SMACH,ITIME,SLØPE,GØSH
1     DIMENSIØN A(3,3),B(3,3),C(3),Q(3),R(3),S(3),T(3),SUM(3),U(61,61,3)
1     DIMENSIØN DELX(1),DELY(1),DELT(1),GAMMA(1),H(1),YØU(1),PØW(1)
1     DIMENSIØN SLØPE(1),WASH(121,2),GØSH(121,2)
      DIMENSIØN SMACH(121)
      DIMENSIØN KØØK(10),GØØK(3)
      EQUIVALENCE (KØØK(1),KØØK1),(KØØK(2),KØØK2),(KØØK(3),KØØK3),(KØØK(
     X4),KØØK4),(KØØK(5),KØØK5),(KØØK(6),KØØK6),(KØØK(7),KØØK7),(KØØK(8)
     X,KØØK8),(KØØK(9),KØØK9),(KØØK(10),KØØK10),(GØØK(2),GØØK2)
1     CØNST1=(2.,0.)*DELX
      CØNST2=2.*DELY
1     CØNST3=DELT/(2.,0.)
      CALL BØRDER(I1,I2,KØØK,GØØK)
      DØ 1 K=1,N
```

```
1     R(K)=U(KØØK5,KØØK10,K)
1     Q(K)=(U(KØØK1,KØØK6,K)-U(KØØK2,KØØK7,K))/CØNST1
      S(K)=(U(KØØK3,KØØK8,K)-U(KØØK4,KØØK9,K))/CØNST2
 1    S(K+3)=(U(KØØK3,KØØK8,K+3)-(GØØK2*U(KØØK4,KØØK9,K+3)))/CØNST2
      CALL ABC(I1,I2,1,1,1,1,KØØK,GØØK)
      CALL MAVE(A,Q,R,B,S,Q)
      DØ 2 K=1,N
12    R(K)=CØNST3*(R(K)+Q(K)+C(K))+U(KØØK5,KØØK10,K)
      CALL ABC(I1,I2,1,1,1,2,KØØK,GØØK)
      RETURN
      END
C
CTEDIUM     DØES NECESSARY BUT BØRING ARITHMETIC FØR SUBRØUTINE AIM
C           CØMPUTES  A*U-SUB-KSI + B*L-SUB-ETA + C  AT (I,J) AND
C           STØRES IT IN ALL
      SUBRØUTINE TEDIUM(I,J,ALL)
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,YØU,STARTX,PØW
      CØMMØN A,B,C,U,Q,R,S,T,SUM,WASH,SMACH,ITIME,SLØPE,GØSH
1     DIMENSIØN A(3,3),B(3,3),C(3),Q(3),R(3),S(3),T(3),SUM(3),U(61,61,3)
1     DIMENSIØN DELX(1),DELY(1),DELT(1),GAMMA(1),H(1),YØU(1),PØW(1)
1     DIMENSIØN SLØPE(1),WASH(121,2),GØSH(121,2)
      DIMENSIØN SMACH(121)
1     DIMENSIØN ALL(3)
      DIMENSIØN L(10),FL(3)
      EQUIVALENCE (L(1),L1),(L(2),L2),(L(3),L3),(L(4),L4),(L(5),L5),(L(6
     X),L6),(L(7),L7),(L(8),L8),(L(9),L9),(FL(1),FL1),(FL(2),FL2),(FL(3)
     X,FL3)
      C1=2.*DELX
      C2=2.*DELY
      CALL BØRDER(I+1,J,L,FL)
      DØ 1 K=1,N
      Q(K)=(U(L1,L6,K)-U(L2,L7,K))/C1
      Q(K+3)=(FL1*(U(L1,L6,K+3)-U(L2,L7,K+3)))/C1
      R(K)=(U(L3,L8,K)-U(L4,L9,K))/C2
 1    R(K+3)=((FL3*U(L3,L8,K+3))-(FL2*U(L4,L9,K+3)))/C2
      CALL ABC(I+1,J,1,1,1,0,L,FL)
      CALL MAVE(A,Q,S,B,R,C)
      DØ 6 K=1,N
16    ALL(K)=S(K)+Q(K)+C(K)
      RETURN
      END
C
CABC        CØMPUTES THE HERMITIAN MATRICES  A = (R + R STAR)/2,
C                      B = (R - R STAR)/2I  AND C AT (L1,L2)
C           SEE EQN (5.4) IN PAPER
C                AND C AT (L1,L2)
C           A IS NØT CØMPUTED IF L3 = 0, IS CØMPUTED IF L3 = 1
C           B IS NØT CØMPUTED IF L4 = 0, IS CØMPUTED IF L4 = 1
C           C IS NØT CØMPUTED IF L5 = 0, IS CØMPUTED IF L5 = 1
      SUBRØUTINE ABC(L1,L2,L3,L4,L5,L6,MACK,TRUCK)
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,YØU,STARTX,PØW
      CØMMØN A,B,C,U,Q,R,S,T,SUM,WASH,SMACH,ITIME,SLØPE,GØSH
1     DIMENSIØN A(3,3),B(3,3),C(3),Q(3),R(3),S(3),T(3),SUM(3),U(61,61,3)
1     DIMENSIØN DELX(1),DELY(1),DELT(1),GAMMA(1),H(1),YØU(1),PØW(1)
1     DIMENSIØN SLØPE(1),WASH(121,2),GØSH(121,2)
      DIMENSIØN SMACH(121)
1     DIMENSIØN TEMP(3),ACØNTR(3,3),BIGA(3,3),XPR(1),YPR(1),TPR(1)
      DIMENSIØN MACK(10),TRUCK(3),NIX(10),FIX(3)
      EQUIVALENCE (NIX(1),NIX1),(NIX(2),NIX2),(NIX(3),NIX3),(NIX(4),NIX4
```



```
     X1,(NIX(5),NIX5),(NIX(6),NIX6),(NIX(7),NIX7),(NIX(8),NIX8),(NIX(9),
     XNIX9),(NIX(10),NIX10),(FIX(1),FIX1),(FIX(2),FIX2),(FIX(3),FIX3)
      CØ 15 I=1,10
15    NIX(I)=MACK(I)
      CØ 16 I=1,3
16    FIX(I)=TRUCK(I)
      CØ 9 K=1,N
      IF(L6) 1,7,2
7     TEMP(K)=U(NIX5,NIX10,K)
      TEMP(K+3)=FIX1*U(NIX5,NIX10,K+3)
      GØ TØ 9
12    TEMP(K)=R(K)
      GØ TØ 9
1     TEMP(K)=(U(NIX1,NIX6,K)+U(NIX2,NIX7,K)+U(NIX3,NIX8,K)+U(NIX4,NIX9,
     XK))/4.
      TEMP(K+3)=((FIX1*(U(NIX1,NIX6,K+3)+U(NIX2,NIX7,K+3)))+(FIX3*U(NIX3
     X,NIX8,K+3))+(FIX2*U(NIX4,NIX9,K+3)))/4.
9     CONTINUE
      IF(L6-1) 8,8,13
8     INK=1
      GØ TØ 14
13    INK=3
14    CALL THE(BIGA,TEMP,L1,L2,L5,INK)
      DØ 10 I=1,N
      DØ 10 J=1,N
1     ACØNTR(I,J)=BIGA(J,I)
10    ACØNTR(I,J+3)=-ACØNTR(I,J+3)
      IF(L3)5,3,4
4     DØ 11 I=1,N
      DØ 11 J=1,N
111   A(I,J)=(BIGA(I,J)+ACØNTR(I,J))/(2.,0.)
3     IF(L4)5,5,6
6     DØ 12 I=1,N
      DØ 12 J=1,N
112   B(I,J)=(BIGA(I,J)-ACØNTR(I,J))/(0.,2.)
5     RETURN
      END
C
CBØRDER     APPLIES REFLECTIØN PRINCIPLE FØR BØUNDARY GRID PØINTS
      SUBRØUTINE BØRDER(M1,M2,NØØK,HØØK)
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,Y2U,STARTX,PØW
      DIMENSIØN LØØK(6),HØØK(3),NØØK(10)
      NY1=NY+1
      LØØK(1)=M1
      LØØK(2)=M1-1
      LØØK(3)=M1+1
      LØØK(4)=M2
      LØØK(5)=M2-1
      LØØK(6)=M2+1
      DØ 1 ING=4,6
      IF(LØØK(ING)) 2,2,3
2     LØØK(ING)=2-LØØK(ING)
      HØØK(ING-3)=-1.0
      GØ TØ 1
3     HØØK(ING-3)=1.0
1     CØNTINUE
      NØØK(1)=LØØK(3)
      NØØK(2)=LØØK(2)
      NØØK(3)=LØØK(1)
```



```
      NØØK(4)=LØØK(1)
      NØØK(5)=LØØK(1)
      NØØK(6)=LØØK(4)
      NØØK(7)=LØØK(4)
      NØØK(8)=LØØK(6)
      NØØK(9)=LØØK(5)
      NØØK(10)=LØØK(4)
      DØ 4 ING=1,5
      MINØØK=NØØK(ING)-NX2
      IF(MINØØK) 4,4,5
5     NØØK(ING)=MINØØK
      NØØK(ING+5)=NY1-NØØK(ING+5)
4     CØNTINUE
      RETURN
      END
C
CCEASE      CØMPUTES THE SPECTRAL RADIUS ØF THE 3X3 MATRIX AND,
C           STØRES IT IN DESIST, TAKING ADVANTAGE ØF THE DEGENERATE
C                  CHARACTER ØF THE MATRIX  AND
      SUBRØUTINE CEASE(AND,DESIST)
1     DIMENSIØN AND(3,3)
      BEE=AND(1,1)+AND(2,2)
      ARG=((BEE*BEE)-(4.*((AND(1,1)*AND(2,2))-(AND(1,2)*AND(1,2))-(AND(1
     X,3)*AND(1,3)))))
      IF(ARG) 1,2,2
1     DESIST=-11111.
      GØ TØ 3
2     RØØT=SQRTF(ARG)
      RØØT1=ABSF((BEE+RØØT)*.5)
      RØØT2=ABSF((BEE-RØØT)*.5)
      RØØT3=ABSF(AND(3,3))
      DESIST=MAX1F(RØØT1,RØØT2,RØØT3)
3     RETURN
      END
C
CMAVE       MATRIX VECTØR MULTIPLIER
      SUBRØUTINE MAVE(Z1,Z2,Z3,Z4,Z5,Z6)
1     DIMENSIØN Z1(3,3),Z2(3),Z3(3),Z4(3,3),Z5(3),Z6(3)
      CALL MAVIE(Z1,Z2,Z3)
      CALL MAVIE(Z4,Z5,Z6)
      RETURN
      END
C
CMAVIE      SCAPEGØAT FØR MAVE  MULTIPLIES MATRIX Q1 BY VECTØR Q2
C           AND STØRES PRØDUCT IN VECTØR Q3
      SUBRØUTINE MAVIE(Q1,Q2,Q3)
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,YØU,STARTX,PØW
1     DIMENSIØN Q1(3,3),Q2(3),Q3(3)
      DØ 2 J=1,N
1     ALL=(0.0,0.0)
      DØ 1 I=1,N
11    ALL=ALL+Q1(J,I)*Q2(I)
12    Q3(J)=ALL
      RETURN
      END
C
CØUTPUT     SPEWS ØUT THE ANSWERS
      SUBRØUTINE ØUTPUT
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,YØU,STARTX,PØW
```



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      CØMMØN A,B,C,U,Q,R,S,T,SUM,WASH,SMACH,ITIME,SLØPE,GØSH
1     DIMENSIØN A(3,3),B(3,3),C(3),Q(3),R(3),S(3),T(3),SUM(3),U(61,61,3)
1     DIMENSIØN DELX(1),DELY(1),DELT(1),GAMMA(1),H(1),Y2U(1),PØW(1)
1     DIMENSIØN SLØPE(1),WASH(121,2),GØSH(121,2)
      DIMENSIØN SMACH(121)
      BUNCH=(FLØATF(ITIME)*DELT)/SQRTF((SLØPE*SLØPE)+1.)
      WRITE ØUTPUT TAPE 6,3
      IF(ITIME-1) 4,5,5
4     INCRE=1
      II=1
      JJ=NX
      GØ TØ 6
5     INCRE=2
      II=ITIME+2
      JJ=NX-1-ITIME
6     DØ 1 I=II,JJ,INCRE
      YY=((FLØATF(I-1)*DELX)+STARTX)-(SLØPE*BUNCH)
      XX=BUNCH+FUNCT(YY)
1     WRITE ØUTPUT TAPE 6,2,XX,YY,(GØSH(I,J),J=1,2),(WASH(I,J),J=1,2),SM
     XACH(I),(GØSH(I,J+2),J=1,2),(WASH(I,J+2),J=1,2)
      RETURN
2     FØRMAT(/(3X,1P7E18.7/39X1P4E18.7))
3     FØRMAT(////14X1HX,17X1HY,17X1HC,17X1HV,16X3HPSI,12X8HPRESSURE,9X11
     XHMACH NUMBER//)
      END
C
CTHE        CØMPUTES CØEFFICIENT MATRIX R AND STØRES IT IN C
C                 SEE EQN (2.2C) IN PAPER
C           ØNLY CØMPUTES PRESSURE AND MACH NUMBER AT (L1,L2) FØR ØUTPUT
C                 WHEN L7 = 2
C           ALSØ CØMPUTES VECTØR S AND STØRES IT IN C IF L5 IS NØNZERØ
      SUBRØUTINE THE(E,TEMP,L1,L2,L5,L7)
      CØMMØN N,NX,NY,NT,NX2,NY2,DELX,DELY,DELT,GAMMA,H,Y2U,STARTX,PØW
      CØMMØN A,B,C,U,Q,R,S,T,SUM,WASH,SMACH,ITIME,SLØPE,GØSH
1     DIMENSIØN A(3,3),B(3,3),C(3),Q(3),R(3),S(3),T(3),SUM(3),U(61,61,3)
1     DIMENSIØN DELX(1),DELY(1),DELT(1),GAMMA(1),H(1),Y2U(1),PØW(1)
1     DIMENSIØN SLØPE(1),WASH(121,2),GØSH(121,2)
      DIMENSIØN SMACH(121)
1     DIMENSIØN E(3,3),TEMP(3),HCAM(1),D(1),C2U2(1),C1(1),FPRAYM(1)
1     DIMENSIØN STATE(1),SØUND(1),USQ(1),VSQ(1),CS(1)
1     BUGGER=SQRTF((SLØPE*SLØPE)+(1.,0.))
      GØ TØ (28,29,30),L7
127   PLANT=FLØATF(ITIME)
      GØ TØ 31
128   PLANT=FLØATF(ITIME-1)
      GØ TØ 31
130   PLANT=(0.5,0.0)
 31   SLANT=(SLØPE*PLANT*DELT)/BUGGER
1     C=TEMP(3)/Y2U
1     FPRAYM=DERIV(D)
1     FPPRYM=DDERIV(D)
      C(1)=((FLØATF(L1-1)*DELX)+STARTX)-SLANT
      C(2)=FLØATF(L2-1)*DELY
1     FPRIME=DERIV(D)
1     C1=(1.,0.)+(FPRAYM*FPRAYM)
      IF(C1(1)) 15,16,15
 16   IF(C1(2)) 15,17,15
 17   WRITE ØUTPUT TAPE 6,18,L1,L2,FPRAYM(1),FPRAYM(2)
      GØ TØ 25
```

```
115    C2=(GAMMA-(1.,0.))/((1.,C.)*GAMMA)
I      C3=(1.,0.)+C2
I      C4=(1.,0.)-C2
I      FUN=(YØU*YØU)/C1
I      C5=(FUN*C4)-C2
       IF(C5(1)) 7,11,7
 11    IF(C5(2)) 7,13,7
 13    WRITE ØUTPUT TAPE 6,14,L1,L2,TEMP(3),TEMP(6),FPRAYM(1),FPRAYM(2)
       GØ TØ 23
17     C6=(C2*FUN)*C3
I      USQ=TEMP(1)*TEMP(1)
I      VSQ=TEMP(2)*TEMP(2)
I      CSQ=USQ+VSQ
I      STATE=EXPF(LØGF(C6/FUN)*GAMMA)*C5
       IF(STATE(1)) 19,20,19
 20    IF(STATE(2)) 19,21,19
 21    WRITE ØUTPUT TAPE 6,22,L1,L2
       GØ TØ 23
119    RHØGAM=((H-(QSQ/(2.,0.)))*(GAMMA-(1.,0.)))/(GAMMA*STATE)
I      SØUND=GAMMA*STATE*RHØGAM
       IF(SØUND(1)) 24,25,24
 25    IF(SØUND(2)) 24,26,24
 26    WRITE ØUTPUT TAPE 6,27,L1,L2,USQ(1),USQ(2),VSQ(1),VSQ(2)
       GØ TØ 23
124    RHØ=EXPF(LØGF(RHØGAM)/(GAMMA-(1.,0.)))
       GØ TØ (12,10,12),L7
I10    WASH(L1,2)=STATE*RHØGAM*RHØ
I      WASH(L1,1)=TEMP(3)
I      HCAM=CSQ/SØUND
       SMACH(L1)=SQRTF(HCAM(1))
I      GØSH(L1,1)=TEMP(1)
I      GØSH(L1,2)=TEMP(2)
       GØ TØ 8
112    C2V2=SØUND-VSQ
I      C2U2=SØUND-USQ
I      YUVI=TEMP(1)*TEMP(2)
I      E(1,2)=-C2V2
I      TERM=-FPRIME*E(1,2)
I      E(2,1)=C2U2
I      C=C2U2+((2.,0.)*YUVI*FPRIME)+(FPRIME*TERM)
I      E(3,3)=-SLØPE/BUGGER
I      E(2,2)=TERM-(SLØPE*D)
I      E(1,1)=E(2,2)+((2.,0.)*YUVI)
       IF(D(1)) 2,3,2
 3     IF(D(2)) 2,4,2
 4     WRITE ØUTPUT TAPE 6,5,(TEMP(I),TEMP(I+3),I=1,3),L1,L2
 23    CALL EXIT
12     DBUG=C*BUGGER
       DØ 1 I=1,2
       DØ 1 J=1,2
11     E(I,J)=E(I,J)/DBUG
       DØ 6 I=1,2
I      E(3,I)=(0.,0.)
I6     E(I,3)=(0.,0.)
       IF(L5) 8,8,9
19     BULK=-((2.,0.)*FPRAYM*FPPRYM*((C4*FUN*C6)-(GAMMA*C3*C5)))/(YØU*C1*
      1C5*C6)
I      QUEUE=(SØUND*BULK*RHØ)/(GAMMA*(GAMMA-(1.,0.)))
I      C(1)=(QUEUE*(YUVI*TERM))/DBUG
```



```
I      C(2)=(QUEUE*((YUVI*FPRIME)+C2U2))/DBUG
I      C(3)=-(TEMP(2)*RHØ)/BUGGER
 8     RETURN
 5     FØRMAT(//10X19HDETERMINANT IS ZERØ/10X6E20.6/10X2I5)
 14    FØRMAT(//10X10HC5 IS ZERØ/10X2I5,4E20.8)
 18    FØRMAT(//10X10HC1 IS ZERØ/10X2I5,2E20.8)
 22    FØRMAT(//10X14HA(PSI) IS ZERØ/10X2I5)
 27    FØRMAT(//10X12HC**2 IS ZERØ/2I5,4E20.8)
       END
C
CDERIV
I      FUNCTIØN DERIV(ARG)
I      DERIV=((3.,0.)*ARG)/SQRTF((1.,0.)+(ARG*ARG))
I      RETURN
       END
C
CDDERIV
I      FUNCTIØN DDERIV(ARG)
I      CØG=(1.,0.)+(ARG*ARG)
I      DENØM=SQRTF(CØG)
I      DDERIV=(3.,0.)*DENØM*((1.,0.)-((ARG*ARG)/CØG))
I      RETURN
       END
C
CFUNCT
       FUNCTIØN FUNCT(ARG)
       FUNCT=3.*SQRTF(1.+(ARG*ARG))
       RETURN
       END
```